XV. *On Systems of Circles and Spheres.*

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INTRODUCTION.

This Memoir is divided into three Parts : Part I. treats of systems of circles in one plane ; Part II. treats of systems of circles on the surface of a sphere ; and Part III. of systems of spheres ; the method of treatment being that indicated in two papers among Clifford's 'Mathematical Papers,' viz., "On Power-Coordinates" (pp. 546–555) and " On the Powers of Spheres" (pp. 332–336). These two papers probably contain the notes of a paper which was read by Clifford to the London Mathematical Society, Feb. 27, 1868, " On Circles and Spheres," which was not published (' Lond. Math. Soc. Proc.,' vol. 2, p. 61). The method of treatment indicated in these papers of Clifford's was successfully applied by the author to prove some theorems given by him in a paper " On the Properties of a Triangle formed by Coplanar Circles" (1885) (' Quarterly Journal of Mathematics,' vol. 21), and then to the extension of those theorems to the case of spheres. But as Clifford's papers contained some suggestions as to the application of the same method to the treatment of Bi-circular Quartics, he was induced to develop these ideas and extend the results to the case of the analogous curves on spheres—called by Professor Cayley Spheri-quadrics—and also of cyclides. It is impossible to say whether, if at all, Clifford was indebted to Darboux for any of the ideas contained in the two papers cited above ; but it is noticeable that they coincide in a great measure with those expressed by Darboux in several papers published during the years 1869–1872.

In Part I. (§§ 1–124) of this Memoir a general relation is first shown to subsist between the powers of any two groups of five circles ; the definition of the power of two circles, as the extension of Steiner's " power of a point and a circle," being due to Darboux, but the definition is here slightly modified so as to include the case when the radius of either (or each) circle is infinite. In Chapter II. an extension of the definition so as to apply to a certain system of conics is given ; this is practically adapted from Chapter II. in Professor Casey's Memoir " On Bicircular Quartics "

(1867) ('Irish Acad. Trans.,' vol. 24). In Chapter III. the general theorem is applied to several interesting cases of circles ; some of the results of this chapter are believed to be new. In Chapter IV. the problem of drawing a circle to cut three given circles at given angles is considered, and the circles connected with a triangle formed by three circles, which are analogous to the circumcircle, the inscribed and escribed, and the nine-points circle of an ordinary triangle are discussed. The results are the same, with one or two exceptions which may be new, as arrived at, but in a different manner, in the paper by the author in the ' Quarterly Journal' (vol. 21). In Chapter V. the power-coordinates of a point (or circle) are defined, and the equations of circles, &c., discussed ; and it is shown that there are two simple coordinate systems of reference ; one consisting of four orthogonal circles, mentioned by CLIFFORD (CASEY and DARBOUX consider five orthogonal spheres), the other consisting of two orthogonal circles and their two points of intersection, which seems to have been indicated for the first time by Mr. HOMERSHAM COX in a paper " On Systems of Circles and Bicircular Quartics" ('Quarterly Journal,' vol. 19, 1883). In Chapter VI. the general equation of the second degree in power-coordinates is discussed, and in Chapter VII. Bi-circular Quartics are classified according to the number of principal circles which they possess. In Chapter VIII. the connexion between Bi-circular Quartics and their focal conics is briefly indicated, the circle of curvature is found, and an expression for the radius of curvature at any point of a bi-circular quartic is investigated. In these last three chapters the results are probably all old, but as the method employed is different from any previously used to discuss these curves in detail, it may not be without interest.

In Part II. (§§ 125–198) almost all the results given in Part I. are extended, with occasionally some slight modifications, to the case of small circles on a sphere and spheri-quadrics.

In Part III. (§§ 199–287) the same order is followed as in Part I.; most of the results in Chapter III., Part I., are extended to the analogous systems of spheres. In Chapter III., however, it is shown that though there is a group of spheres corresponding to the circum-sphere of a tetrahedron, and though several analogous theorems are true for what correspond to the inscribed and escribed spheres, yet there is no analogy to FEUERBACH's theorem. Chapter IV. corresponds exactly to Chapter V. in Part I., and in Chapter V. the general equation of the second degree in power-coordinates is shown to represent a cyclide, and the equation is discussed in the same manner as in Part I., Chapter VI. The reduction, however, of the general equation to its simplest form presents some difficulty. In Chapter VI. cyclides are briefly classified in the order in which they present themselves in reducing the general equation, and in Chapter VII. a few miscellaneous propositions are discussed, as, for instance, the determination of the locus of the centres of the bitangent spheres, *i.e.*, the Focal Quadrics.

It may be convenient to state here the Memoirs consulted :—

CASEY.

" On Bicircular Quartics " (1867), ' Irish Acad. Trans.,' vol. 24.

" On Cyclides and Sphero-Quartics " (1871), ' Phil. Trans.,' vol. 161, pp. 585–721.

CLIFFORD.*

" On the Powers of Spheres" (1868), ' Mathematical Papers.' 1882. Pp. 332–336.

" Of Power-Coordinates in general " (1866), ' Mathematical Papers.' 1882. (Appendix.) Pp. 546–555.

Cox, H. " On Systems of Circles and Bicircular Quartics," ' Quart. Journ. Math.,' vol. 19, 1883, pp. 74–124.

DARBOUX.

" Sur les Relations entre les groupes de Points, de Cercles et de Sphères dans le plan et dans l'espace," ' Annales de l'École Normale Supérieure,' vol. 1, 1872, pp. 323–392.

' Sur une Classe remarquable de Courbes et de Surfaces Algébriques.' Paris, 1873.

SALMON.

' Higher Plane Curves.' 3rd edition, 1879, pp. 240–253.

' Geometry of Three Dimensions.' 4th edition, 1882, pp. 527–536.

PART I.—SYSTEMS OF CIRCLES IN ONE PLANE.

CHAPTER I.—GENERAL SYSTEMS OF CIRCLES.

Definitions. §§ 1–5.

1. The *power* of two circles (or of one circle with respect to the other) is the square of the distance between the centres of the circles, less the sum of the squares of their radii.

Thus denoting the power of two circles whose radii are r_1, r_2 by $\pi_{1,2}$: if $d_{1,2}$ be the distance between their centres, we have

$$\pi_{1,2} = d^2_{1,2} - r_1^2 - r_2^2,$$

or if $\omega_{1,2}$ be the angle at which the circles intersect we have

$$\pi_{1,2} = 2r_1 r_2 \cos \omega_{1,2}.$$

2. If one of the circles reduces to a point, then the power becomes equal to the square of the tangent from the point to the circle. In this case the definition agrees with STEINER's definition of the power of a point with respect to a circle (' CRELLE, Journ. Math.,' vol. 1, 1826, p. 164). The use of the word power is of great antiquity—the area

[* The probable date of these papers is given as 1866 and 1868 respectively. *Cf.* Preface to ' Math. Papers,' pp. xxi, xxii; and also note on page 332.—October, 1886.]

of the parallelogram formed by joining the points, in which two parallel tangents to a hyperbola meet the asymptotes, was called the " Power of the Hyperbola "—and the name was borrowed by STEINER, in the paper quoted above, which was written in 1826, to express the constant rectangle of the segments of any chord of a circle through a point, and this rectangle he called the power of the point with respect to the circle. STEINER also extended his definition thus : if O be one of the centres of similitude of two circles whose centres are A, B : and if a chord through O cut the circles in P and Q respectively, but so that AP, BQ are not parallel, then he proposed to call the rectangle OP, OQ the power of the two circles with respect to O.

DARBOUX seems to have been the first to give the definition of the power of two circles, as used in this memoir, in a paper written in 1872 and published in the 'Annales de l'École Normale Supérieure,' vol. 1. CLIFFORD also gives the same definition in a paper, the probable date of which is said to be 1866, given in the Appendix to his 'Mathematical Papers' (1882); but the paper itself does not seem to have ever been published.

Mr. HOMERSHAM COX, in a paper published in the 'Quarterly Journal of Mathematics' (vol. 19, 1883), has shown that the power of two circles may also be defined as the product of two circles.

3. If the equations of two circles be

$$x^2+y^2+2g_1x+2f_1y+c_1=0, \qquad \text{. . . . . (1)}$$

$$x^2+y^2+2g_2x+2f_2y+c_2=0, \qquad \text{. . . . . (2)}$$

we have

$$\pi_{1,2}=c_1+c_2-2g_1g_2-2f_1f_2. \qquad \text{. . . . . (3)}$$

4. It will be convenient to define the power of a straight line and a circle as twice the perpendicular from the centre of the circle on the line ; and the power of two straight lines as twice the cosine of the angle between them.[*]

Thus the power of (1) and the straight line

$$-2x \cos \alpha - 2y \sin \alpha + 2p = 0, \qquad \text{. . . . . (4)}$$

is given by

$$\pi = 2p + 2g_1 \cos \alpha + 2f_1 \sin \alpha. \qquad \text{. . . . . . . (5)}$$

Thus regarding (4) as a degenerate form of (2), (5) may be considered as a particular case of (3).

[* We can easily show that these definitions are included in that given in § 1. Thus, considering a straight line as a circle of infinite radius, R say, the power of a circle, radius r, with respect to it $=(p+R)^2-r^2-R^2=2pR$, in the limit, p being the perpendicular from the centre of the circle on the straight line. Similarly the power of two straight lines, inclined at an angle ω, $=2R^2 \cos \omega$. Consequently, as we are going to deduce our results from a certain symmetrical determinant, we may ignore these factors R, R^2, and define these powers as in § 4.—22nd October, 1886.]

Again, the power of (4) and

$$-2x \cos \beta - 2y \sin \beta + 2q = 0,$$

is given by

$$\pi = -2 \cos \beta \, (\alpha - \beta) = -2 \cos \alpha \cos \beta - 2 \sin \alpha \sin \beta \, ;$$

which may also be considered as a particular case of (3).

The equation to the line at infinity may be written

$$0x + 0y + 1 = 0.$$

Hence, denoting this line by the symbol θ, we shall have

$$\pi_{\theta, x} = 1, \text{ if } x \text{ denote any circle, or point,}$$

and

$$\pi_{\theta, x} = 0, \text{ if } x \text{ denote any straight line,}$$

and

$$\pi_{\theta, \theta} = 0, \text{ of course.}$$

5. It will be convenient to observe here, that if π' denote the power of the two circles which are respectively the inverse curves of (1) and (2) with respect to any circle, whose centre is the origin O and radius R ; then

$$\pi' = \frac{R^4}{c_1 c_2} \pi \, ;$$

i.e., denoting the circles by S_1, S_2, and the circles inverse to them by S'_1, S'_2, since $c_1 = \pi_{0, s_1}$, $c_2 = \pi_{0, s_2}$,

$$\frac{\pi_{s'_1, s'_2}}{\sqrt{\pi_{0, s'_1} \pi_{0, s'_2}}} = \frac{\pi_{s_1, s_2}}{\sqrt{\pi_{0, s_1} . \pi_{0, s_2}}},$$

and the formula is still true if either or both circles degenerate into straight lines.

Thus if x, y denote any circles, straight lines, or points, the expresssion

$$\frac{\pi_{x, y}}{\sqrt{\pi_{0, x} . \pi_{0, y}}}$$

is unaltered if the circles be inverted with respect to any circle whose centre is O.

General Theorems. §§ 6–8.

6. If we have given a system of five circles (1, 2, 3, 4, 5), their powers with respect to any five other circles (6, 7, 8, 9, 10) are connected by identical relation, which may be expressed in the umbral notation by

$$\Pi \begin{pmatrix} 1, 2, 3, 4, 5 \\ 6, 7, 8, 9, 10 \end{pmatrix} = 0.$$

The word "circle" being intended to include a point, a straight line, or the line at infinity.

This is easily proved by multiplying together the two matrices

$$
\begin{vmatrix}
1, & 2g_1, & 2f_1, & c_1 \\
1, & 2g_2, & 2f_2, & c_2 \\
1, & 2g_3, & 2f_3, & c_3 \\
1, & 2g_4, & 2f_4, & c_4 \\
1, & 2g_5, & 2f_5, & c_5
\end{vmatrix}
\qquad
\begin{vmatrix}
c_6, & -g_6, & -f_6, & 1 \\
c_7, & -g_7, & -f_7, & 1 \\
c_8, & -g_8, & -f_8, & 1 \\
c_9, & -g_9, & -f_9, & 1 \\
c_{10}, & -g_{10}, & -f_{10}, & 1
\end{vmatrix}
$$

and we obtain at once the equation

$$
\begin{vmatrix}
\pi_{1,6}, & \pi_{1,7}, & \pi_{1,8}, & \pi_{1,9}, & \pi_{1,10} \\
\pi_{2,6}, & \pi_{2,7}, & \pi_{2,8}, & \pi_{2,9}, & \pi_{2,10} \\
\pi_{3,6}, & \pi_{3,7}, & \pi_{3,8}, & \pi_{3,9}, & \pi_{3,10} \\
\pi_{4,6}, & \pi_{4,7}, & \pi_{4,8}, & \pi_{4,9}, & \pi_{4,10} \\
\pi_{5,6}, & \pi_{5,7}, & \pi_{5,8}, & \pi_{5,9}, & \pi_{5,10}
\end{vmatrix} = 0, \quad \ldots \ldots \quad (6)
$$

which may be conveniently written

$$
\Pi\begin{pmatrix} 1, 2, 3, 4, 5 \\ 6, 7, 8, 9, 10 \end{pmatrix} = 0. \quad \ldots \ldots \ldots \quad (7)
$$

7. An important particular case is when θ is a member of both systems : then we have

$$
\Pi\begin{pmatrix} \theta, 1, 2, 3, 4 \\ \theta, 5, 6, 7, 8 \end{pmatrix} = 0, \quad \ldots \ldots \ldots \quad (8)
$$

or

$$
\begin{vmatrix}
0, & 1, & 1, & 1, & 1 \\
1, & \pi_{1,5}, & \pi_{1,6}, & \pi_{1,7}, & \pi_{1,8} \\
1, & \pi_{2,5}, & \pi_{2,6}, & \pi_{2,7}, & \pi_{2,8} \\
1, & \pi_{3,5}, & \pi_{3,6}, & \pi_{3,7}, & \pi_{3,8} \\
1, & \pi_{4,5}, & \pi_{4,6}, & \pi_{4,7}, & \pi_{4,8}
\end{vmatrix} = 0.
$$

Whence, denoting the angle of intersection of the two circles (x, y) by $\omega_{x,y}$, we have, provided none of the circles reduce to points,

$$
\begin{vmatrix}
0, & \dfrac{1}{r_5}, & \dfrac{1}{r_6}, & \dfrac{1}{r_7}, & \dfrac{1}{r_8} \\
\dfrac{1}{r_1}, & \cos \omega_{1,5}, & \cos \omega_{1,6}, & \cos \omega_{1,7}, & \cos \omega_{1,8} \\
\dfrac{1}{r_2}, & \cos \omega_{2,5}, & \cos \omega_{2,6}, & \cos \omega_{2,7}, & \cos \omega_{2,8} \\
\dfrac{1}{r_3}, & \cos \omega_{3,5}, & \cos \omega_{3,6}, & \cos \omega_{3,7}, & \cos \omega_{3,8} \\
\dfrac{1}{r_4}, & \cos \omega_{4,5}, & \cos \omega_{4,6}, & \cos \omega_{4,7}, & \cos \omega_{4,8}
\end{vmatrix} = 0. \quad \ldots \quad (9)
$$

This is true if any of the circles are replaced by straight lines.

If we take the two systems (1, 2, 3, 4), (5, 6, 7, 8) as coincident, we have an equation which gives us the radii of the two circles which cut three given circles at given angles.

Thus, if three given circles, radii r_1, r_2, r_3 cut at angles α, β, γ, the radii of the circles which cut them at angles ϕ_1, ϕ_2, ϕ_3 respectively, are the roots of the equation

$$\begin{vmatrix} 0, & \dfrac{1}{r_1}, & \dfrac{1}{r_2}, & \dfrac{1}{r_3}, & \dfrac{1}{\rho}, \\[2ex] \dfrac{1}{r_1}, & -1, & \cos\gamma, & \cos\beta, & \cos\phi_1 \\[2ex] \dfrac{1}{r_2}, & \cos\gamma, & -1, & \cos\alpha, & \cos\phi_2 \\[2ex] \dfrac{1}{r_3}, & \cos\beta, & \cos\alpha, & -1, & \cos\phi_3 \\[2ex] \dfrac{1}{\rho}, & \cos\phi_1, & \cos\phi_2, & \cos\phi_3, & -1 \end{vmatrix} = 0. \quad\quad\quad (10)$$

8. Another important theorem is easily deduced by the method employed in § 6, thus

$$\Pi\begin{pmatrix} 1, 2, 3, 4 \\ 5, 6, 7, 8 \end{pmatrix} = \begin{vmatrix} 1, & 2g_1, & 2f_1, & c_1 \\ 1, & 2g_2, & 2f_2, & c_2 \\ 1, & 2g_3, & 2f_3, & c_3 \\ 1, & 2g_4, & 2f_4, & c_4 \end{vmatrix} \times \begin{vmatrix} c_5, & -g_5, & -f_5, & 1 \\ c_6, & -g_6, & -f_6, & 1 \\ c_7, & -g_7, & -f_7, & 1 \\ c_8, & -g_8, & -f_8, & 1 \end{vmatrix} ;$$

hence we have at once

$$\left\{ \Pi\begin{pmatrix} 1, 2, 3, 4 \\ 5, 6, 7, 8 \end{pmatrix} \right\}^2 = \Pi\begin{pmatrix} 1, 2, 3, 4 \\ 1, 2, 3, 4 \end{pmatrix} \times \Pi\begin{pmatrix} 5, 6, 7, 8 \\ 5, 6, 7, 8 \end{pmatrix}. \quad\quad\quad (11)$$

CHAPTER II.—EXTENSION OF RESULTS OF CHAPTER I.

Preliminary Remarks.—§§ 9–12.

9. Dr. CASEY has shown in his memoir "On Bicircular Quartics,' that any two conics whose equations can be put in the form $S - L^2 = 0$, and $S - M^2 = 0$, possess a pair of angles which he calls their anharmonic angles, and which he shows to be analogous to the angle of intersection of two circles. Thus, if S', S'' denote the results of substituting the coordinates of the poles of L, M respectively in S, and R

the result of substituting the coordinates of the pole of L in M, then the anharmonic angles θ, ϕ of the two conics are defined by :—

$$1 - R = \sqrt{(1-S')(1-S'')}.\cos\theta,$$

$$1 + R = \sqrt{(1-S')(1-S'')}.\cos\phi.$$

A proof of a theorem similar to that given in § 6 of the present memoir is to be found in SALMON's 'Conics,' p. 366.

10. Let us take as the equations of the three conics

$$S \equiv x^2 + y^2 + z^2 = 0,$$

$$a^2(x^2 + y^2 + z^2) - (xf + yg + zh)^2 = 0,$$

$$a'^2(x^2 + y^2 + z^2) - (xf' + yg' + zh')^2 = 0.$$

Then following the method used in CASEY's paper (§ 126), we form the discriminant of

$$aS^{\frac{1}{2}} \pm (xf + yg + zh) + \lambda\{a'S^{\frac{1}{2}} \pm (xf' + yg' + zh')\} = 0 ;$$

and we obtain

$$(a'^2 - S')\lambda^2 + 2(aa' - R)\lambda + a^2 - S = 0 ;$$

and so, if we take

$$aa' - R = \sqrt{(a'^2 - S')(a^2 - S)}.\cos\theta,$$

where $R = ff' + gg' + hh'$, $S = f^2 + g^2 + h^2$, $S' = f'^2 + g'^2 + h'^2$; we have for the tact-invariant of the conics $a^2S - L^2$, $a'^2S - M^2$, $(a^2 - S)(a'^2 - S')\sin^2\theta = 0$; or $\theta = 0$.

11. Again, forming the discriminant of

$$aS^{\frac{1}{2}} \pm (xf + yg + zh) + \lambda\{a'S \mp (xf' + yg' + zh')\} = 0,$$

we obtain for the tact-invariant, $\phi = 0$, where

$$aa' + R = \sqrt{(a'^2 - S')(a^2 - S)} \cos\phi.$$

12. Either of these expressions, $aa' \pm R$, might be defined as the power of the two conics $a^2S - L^2$, $a'^2S - M^2$. For if $\theta = \frac{\pi}{2}$, it is evident, as CASEY has shown, that the pencil formed by the lines L, M, and the chords of contact of the two line pairs which can be drawn to touch S from the points of intersection of $a'^2L - a^2M$ with $a^2S - L^2$, is harmonic; and so $\theta = \frac{\pi}{2}$ is the condition corresponding to the case of two circles cutting orthogonally. In this case the power vanishes.

Let us define, then, the power of the two conics a^2S-L, and a'^2S-M^2, as the expression $aa'+R$, and let us denote this by π. Thus

$$\pi = aa' + ff' + gg' + hh' ;$$

the conic S being supposed reduced to its standard form, and f, g, h being the co-ordinates of the pole of L with respect to S.

General Theorem.—§§ 13–15.

13. If we have any two systems of conics, say (1, 2, 3, 4, 5), (6, 7, 8, 9, 10), inscribed in the same conic S, the powers of the conics of one system with respect to the conics of the other system are connected by the relation

$$\Pi\begin{pmatrix} 1, 2, 3, 4, 5 \\ 6, 7, 8, 9, 10 \end{pmatrix} = 0, \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (12)$$

i.e., the same equation as in § 6.

Thus taking the same equations as in § 10, by multiplying the matrices

$$\begin{vmatrix} a_1, & f_1, & g_1, & h_1 \\ a_2, & f_2, & g_2, & h_2 \\ a_3, & f_3, & g_3, & h_3 \\ a_4, & f_4, & g_4, & h_4 \\ a_5, & f_5, & g_5, & h_5 \end{vmatrix} \qquad \begin{vmatrix} a_6, & f_6, & g_6, & h_6 \\ a_7, & f_7, & g_7, & h_7 \\ a_8, & f_8, & g_8, & h_8 \\ a_9, & f_9, & g_9, & h_9 \\ a_{10}, & f_{10}, & g_{10}, & h_{10} \end{vmatrix} ,$$

we have at once the relation

$$\begin{vmatrix} \pi_{1,6}, & \pi_{1,7}, & \pi_{1,8}, & \pi_{1,9}, & \pi_{1,10} \\ \pi_{2,6}, & \pi_{2,7}, & \pi_{2,8}, & \pi_{2,9}, & \pi_{2,10} \\ \pi_{3,6}, & \pi_{3,7}, & \pi_{3,8}, & \pi_{3,9}, & \pi_{3,10} \\ \pi_{4,6}, & \pi_{4,7}, & \pi_{4,8}, & \pi_{4,9}, & \pi_{4,10} \\ \pi_{5,6}, & \pi_{5,7}, & \pi_{5,8}, & \pi_{5,9}, & \pi_{5,10} \end{vmatrix} = 0. \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (13)$$

14. This equation has been proved for two systems of conics inscribed in the same conic. The result is also true if any of the conics be replaced by straight lines, provided that we define the power of a straight line and a conic of the system to be the power of the straight line and the chord of contact of the conic and the conic S ; the power of two straight lines being defined as the perpendicular distance from the pole of one line with respect to S to the other line. Thus let any conic of the system be

$$u \equiv a^2S - (fx + gy + hz)^2 = 0,$$

3 R

and any straight line
$$\alpha = lx + my + nz = 0 \; ;$$

then the power of the line and conic is $lf + mg + nh$, and the power of this straight line and the line
$$\alpha' = l'x + m'y + n'z = 0,$$

is $ll' + mm' + nn'$.

Again, we must define the power of S and u to be a, and the power of S and α to be zero; the power of S with respect to S to be unity.

15. Exactly as in § 8 we can show that for any two systems of four conics inscribed in S,

$$\left\{ \Pi \begin{pmatrix} 1, & 2, & 3, & 4 \\ 5, & 6, & 7, & 8 \end{pmatrix} \right\}^2 = \left\{ \Pi \begin{pmatrix} 1, & 2, & 3, & 4 \\ 1, & 2, & 3, & 4 \end{pmatrix} \right\} \times \left\{ \Pi \begin{pmatrix} 5, & 6, & 7, & 8 \\ 5, & 6, & 7, & 8 \end{pmatrix} \right\}.$$

CHAPTER III.— SPECIAL SYSTEMS OF CIRCLES.

Circles Touching three Straight Lines.—§§ 16, 17.

16. Denoting the four circles which touch the sides of a triangle by (1, 2, 3, 4), and the sides of the triangle by a, b, c, we have, if x denote any other circle,

$$\Pi \begin{pmatrix} \theta, & a, & b, & c, & x \\ \theta, & 1, & 2, & 3, & 4 \end{pmatrix} = 0,$$

i.e.,

$$\begin{vmatrix} 0, & 1, & 1, & 1, & 1 \\ 0, & \pi_{a,1}, & \pi_{a,2}, & \pi_{a,3}, & \pi_{a,4} \\ 0, & \pi_{b,1}, & \pi_{b}, & \pi_{b,3}, & \pi_{b,4} \\ 0, & \pi_{c,1}, & \pi_{c,2}, & \pi_{c,3}, & \pi_{c,4} \\ 1, & \pi_{x,1}, & \pi_{x,2}, & \pi_{x,3}, & \pi_{x,4} \end{vmatrix} = 0 \; ;$$

and since
$$\pi_{a,1} = r_1, \; \pi_{a,2} = -r_2, \; \&c.,$$

we have at once
$$\frac{1}{r_1} = \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4}.$$

This theorem will be subsequently extended.

17. Again let x denote the nine-points circle of the triangle, z the inscribed circle, and let (1, 2, 3) denote the mid-points of the sides; then since

$$\pi_{x,1} = \pi_{x,2} = \pi_{x,3} = 0 \; ;$$

$$\pi_{z,1} = (b-c)^2 \; ; \quad \pi_{z,2} = (c-a)^2 \; ; \quad \pi_{z,3} = (a-b)^2 \; ;$$

the equation

$$\Pi\begin{pmatrix} x, & z, & 1, & 2, & 3 \\ x, & z, & 1, & 2, & 3 \end{pmatrix} = 0,$$

gives us

$$\begin{vmatrix} \pi_{x,x}, & \pi_{x,z}, & 0, & 0, & 0 \\ \pi_{z,x}, & \pi_{z,z}, & (b-c)^2, & (c-a)^2, & (a-b)^2 \\ 0, & (b-c)^2, & 0, & \dfrac{c^2}{4}, & \dfrac{b^2}{4} \\ 0, & (c-a)^2, & \dfrac{c^2}{4}, & 0, & \dfrac{a^2}{4} \\ 0, & (a-b)^2, & \dfrac{b^2}{4}, & \dfrac{a^2}{4}, & 0 \end{vmatrix} = 0\,;$$

whence we have

$$\pi_{x,x}.\pi_{z,z} = \pi^2_{x,z}.$$

or the circle which passes through the mid-points of the sides of a triangle touches the inscribed circle. Similarly it can be proved to touch the escribed circles.

Circle Cutting three Circles Orthogonally.—§§ 18–20.

18. Let the circle cutting the system (1, 2, 3) orthogonally be denoted by (x) : then since

$$\Pi\begin{pmatrix} \theta, & x, & 1, & 2, & 3 \\ \theta, & x, & 1, & 2, & 3 \end{pmatrix} = 0\,;$$

we have

$$\pi_{x,x}.\Pi\begin{pmatrix} \theta, & 1, & 2, & 3 \\ \theta, & 1, & 2, & 3 \end{pmatrix} = \Pi\begin{pmatrix} 1, & 2, & 3 \\ 1, & 2, & 3 \end{pmatrix}. \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (14)$$

But

$$\Pi\begin{pmatrix} \theta, & 1, & 2, & 3 \\ \theta, & 1, & 2, & 3 \end{pmatrix} = \begin{vmatrix} 0, & 1, & 1, & 1 \\ 1, & \pi_{1,1}, & \pi_{1,2}, & \pi_{1,3} \\ 1, & \pi_{2,1}, & \pi_{2,2}, & \pi_{2,3} \\ 1, & \pi_{3,1}, & \pi_{3,2}, & \pi_{3,3} \end{vmatrix}\,;$$

and if the equations of the circles be

$$x^2 + y^2 + 2g_r x + 2f_r y + c_r = 0,$$

where $r = (1, 2, 3)$;

we have at once

$$\Pi\begin{pmatrix} \theta, 1, 2, 3 \\ \theta, 1, 2, 3 \end{pmatrix} = \begin{vmatrix} 0, & 0, & 0, & 1 \\ 1, & 2g_1, & 2f_1, & c_1 \\ 1, & 2g_2, & 2f_2, & c_2 \\ 1, & 2g_3, & 2f_3, & c_3 \end{vmatrix} \times \begin{vmatrix} 1, & 0, & 0, & 0 \\ c_1, & -g_1, & -f_1, & 1 \\ c_2, & -g_2, & -f_2, & 1 \\ c_3, & -g_3, & -f_3, & 1 \end{vmatrix}$$

$$= -16\{\Delta(1, 2, 3)\}^2 ; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

where $\Delta(1, 2, 3)$ denotes the area of the triangle formed by the centres of the circles (1, 2, 3).

Hence by (14) we see that, if r denote the radius of the circle which cuts (1, 2, 3) orthogonally,

$$r^2 = \frac{\Pi\begin{pmatrix} 1, 2, 3 \\ 1, 2, 3 \end{pmatrix}}{32\{\Delta(1, 2, 3)\}^2} = \frac{r_1^2 r_2^2 r_3^2}{4\{\Delta(1, 2, 3)\}^2} \times \begin{vmatrix} -1, & \cos \omega_{1,2}, & \cos \omega_{1,3} \\ \cos \omega_{2,1}, & -1, & \cos \omega_{2,3} \\ \cos \omega_{1,3}, & \cos \omega_{2,3}, & -1 \end{vmatrix} ;$$

or

$$r^2 = \frac{r_1^2 r_2^2 r_3^2}{\{\Delta(1, 2, 3)\}^2} \cos s . \cos (s - \omega_{2,3}) . \cos (s - \omega_{3,1}) . \cos (s - \omega_{1,2}) ; \quad . \quad . \quad (16)$$

$\omega_{2,3}, \omega_{3,1}, \omega_{1,2}$ being the angles of intersection of the circles (1, 2, 3), and $2s$ being equal to $\omega_{2,3} + \omega_{3,1} + \omega_{1,2}$.

19. Since any point on a circle may be considered as a circle of infinitely small radius cutting the circle orthogonally, it follows that if three circles meet in a point, the radius of their orthogonal circle must vanish. Hence the condition that three circles meet in a point is

$$\Pi\begin{pmatrix} 1, 2, 3 \\ 1, 2, 3 \end{pmatrix} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

20. The radius of the orthogonal circle of the system (1, 2, 3) is infinite when $\Delta(1, 2, 3) = 0$, $i.e.$, when the centres of the circles lie on a straight line; in which case the orthogonal circle degenerates into the straight line through their centres.

Four circles having a Common Orthogonal Circle.—§§ 21–24.

21. Suppose that the system (1, 2, 3, 4) has a common orthogonal circle, x say; then since

$$\Pi\begin{pmatrix} x, 1, 2, 3, 4 \\ y, 1, 2, 3, 4 \end{pmatrix} = 0 ;$$

we have

$$\begin{vmatrix} \pi_{x,y}, & \pi_{y,1}, & \pi_{y,2}, & \pi_{y,3}, & \pi_{y,4} \\ 0, & \pi_{1,1}, & \pi_{1,2}, & \pi_{1,3}, & \pi_{1,4} \\ 0, & \pi_{2,1}, & \pi_{2,2}, & \pi_{2,3}, & \pi_{2,4} \\ 0, & \pi_{3,1}, & \pi_{3,2}, & \pi_{3,3}, & \pi_{3,4} \\ 0, & \pi_{4,1}, & \pi_{4,2}, & \pi_{4,3}, & \pi_{4,4} \end{vmatrix} = 0 \; ;$$

whence

$$\Pi \begin{pmatrix} 1, 2, 3, 4 \\ 1, 2, 3, 4 \end{pmatrix} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

which is the condition that the system has a common orthogonal circle.

22. It is evident also that $(5, 6, 7, 8)$ being any other system of circles, we must have, since

$$\Pi \begin{pmatrix} x, 5, 6, 7, 8 \\ y, 1, 2, 3, 4 \end{pmatrix} = 0,$$

$$\Pi \begin{pmatrix} 1, 2, 3, 4 \\ 5, 6, 7, 8 \end{pmatrix} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

It follows by symmetry that this result must be true if either of the two systems $(1, 2, 3, 4)$ or $(5, 6, 7, 8)$ have a common orthogonal circle.

Hence we must have

$$\left\{ \Pi \begin{pmatrix} 1, 2, 3, 4 \\ 5, 6, 7, 8 \end{pmatrix} \right\}^2 = \Pi \begin{pmatrix} 1, 2, 3, 4 \\ 1, 2, 3, 4 \end{pmatrix} \times \Pi \begin{pmatrix} 5, 6, 7, 8 \\ 5, 6, 7, 8 \end{pmatrix}.$$

23. We may notice that any three circles, whose centres lie on a straight line, may be considered as having, with the line at infinity, a common orthogonal circle.

24. The system $(1, 2, 3, 4)$ having a common orthogonal circle, and $(5, 6, 7, 8)$ being any other system of circles, we have

$$\Pi \begin{pmatrix} 1, 2, 3, 4 \\ 5, 6, 7, 8 \end{pmatrix} = 0 \; ;$$

and, as in § 8, we may prove that

$$\left\{ \Pi \begin{pmatrix} 1, 2, 3 \\ 5, 6, 7 \end{pmatrix} \right\}^2 = \Pi \begin{pmatrix} 1, 2, 3 \\ 1, 2, 3 \end{pmatrix} \times \Pi \begin{pmatrix} 5, 6, 7 \\ 5, 6, 7 \end{pmatrix}. \quad . \quad . \quad . \quad . \quad . \quad (20)$$

As a particular case, we have

$$\Pi \begin{pmatrix} x, 1, 2, 3 \\ 1, 2, 3, 4 \end{pmatrix} = 0 \; ;$$

whence

$$\pi_{x,1} \cdot \Pi \begin{pmatrix} 1, 2, 3 \\ 2, 3, 4 \end{pmatrix} + \pi_{x,2} \cdot \Pi \begin{pmatrix} 1, 2, 3 \\ 1, 4, 3 \end{pmatrix} + \pi_{x,3} \cdot \Pi \begin{pmatrix} 1, 2, 3 \\ 1, 2, 4 \end{pmatrix} - \pi_{x,4} \cdot \Pi \begin{pmatrix} 1, 2, 3 \\ 1, 2, 3 \end{pmatrix} = 0.$$

Hence by (20)

$$\pi_{x,\,4}\cdot\left\{\Pi\binom{1,\,2,\,3}{1,\,2,\,3}\right\}^{\frac{1}{2}}=\pi_{x,\,1}\cdot\left\{\Pi\binom{2,\,3,\,4}{2,\,3,\,4}\right\}^{\frac{1}{2}}+\pi_{x,\,2}\cdot\left\{\Pi\binom{1,\,4,\,3}{1,\,4,\,3}\right\}^{\frac{1}{2}}+\pi_{x,\,3}\cdot\left\{\Pi\binom{1,\,2,\,4}{1,\,2,\,4}\right\}^{\frac{1}{2}}.$$

But if r be the radius of the common orthogonal circle, we have by (16),

$$r^{2}=\frac{\Pi\binom{1,\,2,\,3}{1,\,2,\,3}}{32\,\{\Delta(1,\,2,\,3)\}^{2}}=\frac{\Pi\binom{1,\,2,\,4}{1,\,2,\,4}}{32\,\{\Delta(1,\,2,\,4)\}^{2}}=\&\text{c}.$$

Therefore

$$\pi_{x,\,4}\cdot\Delta(1,\,2,\,3)=\pi_{x,\,1}\cdot\Delta(2,\,3,\,4)+\pi_{x,\,2}\cdot\Delta(1,\,4,\,3)+\pi_{x,\,3}\cdot\Delta(1,\,2,\,4). \quad . \quad . \quad (21)$$

Thus if any four circles have a common orthogonal circle, and the areal coordinates of the centre of one of them referred to the triangle formed by joining the centres of the other three be α, β, γ : then the powers of any other circle with respect to these four circles are connected by the relation

$$\pi_{x,\,4}=\alpha.\pi_{x,\,1}+\beta.\pi_{x,\,2}+\gamma.\pi_{x,\,3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

As a particular case we obtain the well-known theorem that if A, B, C be the centres of any three circles, P any point on the circle which cuts these orthogonally, and O any other point, then

$$\text{OP}^{2}=(\text{OA}^{2}-r_{1}^{2})\alpha+(\text{OB}^{2}-r_{2}^{2})\beta+(\text{OC}^{2}-r_{3}^{2})\gamma\,;$$

where α, β, γ are the areal coordinates of P referred to the triangle ABC.

Orthogonal Systems.—§§ 25–29.

25. Four circles may be said to form an orthogonal system if they cut one another orthogonally: it is clear that the centre of any one of them is the orthocentre of the triangle formed by the other three.

If the system be denoted by (1, 2, 3, 4), then (x, y) being any other circles, the equation

$$\Pi\binom{x,\,1,\,2,\,3,\,4}{y,\,1,\,2,\,3,\,4}=0$$

becomes

$$\begin{vmatrix} \pi_{x,y}, & \pi_{x,1}, & \pi_{x,2}, & \pi_{x,3}, & \pi_{x,4} \\ \pi_{y,1}, & \pi_{1,1}, & 0, & 0, & 0 \\ \pi_{y,2}, & 0, & \pi_{2,2}, & 0, & 0 \\ \pi_{y,3}, & 0, & 0, & \pi_{3,3}, & 0 \\ \pi_{y,4}, & 0, & 0, & 0, & \pi_{4,4} \end{vmatrix}=0\,;$$

i.e.,

$$\pi_{x,y} = \frac{\pi_{x,1}.\pi_{y,1}}{\pi_{1,1}} + \frac{\pi_{x,2}.\pi_{y,2}}{\pi_{2,2}} + \frac{\pi_{x,3}.\pi_{y,3}}{\pi_{3,3}} + \frac{\pi_{x,4}.\pi_{y,4}}{\pi_{4,4}}. \qquad \ldots \quad (23)$$

As particular cases, we have, since $\pi_{1,1} = -2r_1^2$, &c.,

$$\frac{\pi_{x,1}}{r_1^2} + \frac{\pi_{x,2}}{r_2^2} + \frac{\pi_{x,3}}{r_3^2} + \frac{\pi_{x,4}}{r_4^2} = -2.\pi_{x,\theta} = -2 \text{ or } 0 ; \quad \ldots \quad (24)$$

according as x denotes a circle or a straight line.

If x represents a circle, radius r_x, we have

$$4r_x^2 = \frac{\pi_{x,1}^2}{r_1^2} + \frac{\pi_{x,2}^2}{r_2^2} + \frac{\pi_{x,3}^2}{r_3^2} + \frac{\pi_{x,4}^2}{r_4^2}. \qquad \ldots \quad (25)$$

And if x represent a straight line, so that $\pi_{x,x} = -2$, we have

$$4 = \frac{\pi_{x,1}^2}{r_1^2} + \frac{\pi_{x,2}^2}{r_2^2} + \frac{\pi_{x,3}^2}{r_3^2} + \frac{\pi_{x,4}^2}{r_4^2}. \qquad \ldots \quad (26)$$

Again, since $\pi_{\theta,\theta} = 0$, we have

$$0 = \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} + \frac{1}{r_4^2} ; \qquad \ldots \quad (27)$$

whence it appears that one at least of the three circles must be imaginary, and one at least real.

26. By equation (16) we have

$$r_4^2 = -\frac{r_1^2 r_2^2 r_3^2}{4\{\Delta(1,\,2,\,3)\}^2}$$

Hence

$$4\{\Delta(1,\,2,\,3)\}^2 = r_2^2 r_3^2 + r_3^2 r_1^2 + r_1^2 r_2^2 ; \qquad \ldots \quad (28)$$

and we can easily find that, if ρ be the radius of the nine-points circle of the triangle formed by the centres of three of the circles,

$$(r_1^2 + r_2^2)(r_1^2 + r_3^2)(r_1^2 + r_4^2)(r_2^2 + r_3^2)(r_2^2 + r_4^2)(r_3^2 + r_4^2) + 256.\rho^4 r_1^2 r_2^2 r_3^2 r_4^2 = 0. \quad (29)$$

27. If the circles (1, 2, 3, 4) be any system not having a common orthogonal circle, we may find four other circles, (5, 6, 7, 8) say, each of which is orthogonal to three of the former. Two such systems are connected by several interesting formulæ, and one system may be called the "orthogonal system" of the other.

Thus, x and y denoting any two circles, we have, since

$$\Pi\begin{pmatrix} x, 1, 2, 3, 4 \\ y, 5, 6, 7, 8 \end{pmatrix} = 0.$$

the relation

$$
\begin{vmatrix}
\pi_{x,y}, & \pi_{x,5}, & \pi_{x,6}, & \pi_{x,7}, & \pi_{x,8} \\
\pi_{y,1}, & \pi_{1,5}, & 0, & 0, & 0 \\
\pi_{y,2}, & 0, & \pi_{2,6}, & 0, & 0 \\
\pi_{y,3}, & 0, & 0, & \pi_{3,7}, & 0 \\
\pi_{y,4}, & 0, & 0, & 0, & \pi_{4,8}
\end{vmatrix} = 0 ;
$$

which may be written

$$
\pi_{x,y} = \frac{\pi_{x,5}\cdot\pi_{y,1}}{\pi_{1,5}} + \frac{\pi_{x,6}\cdot\pi_{y,2}}{\pi_{2,6}} + \frac{\pi_{x,7}\cdot\pi_{y,3}}{\pi_{3,7}} + \frac{\pi_{x,8}\cdot\pi_{y,4}}{\pi_{4,8}} ; \quad \cdots \quad (30)
$$

whence as particular cases, we have, x denoting any circle whose radius is r_x—

$$
\frac{\pi_{x,5}}{\pi_{1,5}} + \frac{\pi_{x,6}}{\pi_{2,6}} + \frac{\pi_{x,7}}{\pi_{3,7}} + \frac{\pi_{x,8}}{\pi_{4,8}} = 1, \quad \cdots \quad (31)
$$

$$
\frac{\pi_{x,5}\cdot\pi_{x,1}}{\pi_{1,5}} + \frac{\pi_{x,6}\cdot\pi_{x,2}}{\pi_{2,6}} + \frac{\pi_{x,7}\cdot\pi_{x,3}}{\pi_{3,7}} + \frac{\pi_{x,8}\cdot\pi_{x,4}}{\pi_{4,8}} = -2r_x^2. \quad \cdots \quad (32)
$$

If x denote any straight line, we have

$$
\frac{\pi_{x,5}}{\pi_{1,5}} + \frac{\pi_{x,6}}{\pi_{2,6}} + \frac{\pi_{x,7}}{\pi_{3,7}} + \frac{\pi_{x,8}}{\pi_{4,8}} = 0, \quad \cdots \quad (33)
$$

$$
\frac{\pi_{x,5}\cdot\pi_{x,1}}{\pi_{1,5}} + \frac{\pi_{x,6}\cdot\pi_{x,2}}{\pi_{2,6}} + \frac{\pi_{x,7}\cdot\pi_{x,3}}{\pi_{3,7}} + \frac{\pi_{x,8}\cdot\pi_{x,4}}{\pi_{4,8}} = -2, \quad \cdots \quad (34)
$$

and

$$
\frac{1}{\pi_{1,5}} + \frac{1}{\pi_{2,6}} + \frac{1}{\pi_{3,7}} + \frac{1}{\pi_{4,8}} = \pi_{\theta,\theta} = 0. \quad \cdots \quad (35)
$$

28. This last result gives us an interesting theorem—in the case when the given system (1, 2, 3, 4) consists of four points; then $\pi_{1,5}$ is equal to the square of the tangent from the point (1) to the circle passing through the points (2, 3, 4), and so on; thus the sum of the reciprocals of the powers of each of four given points with respect to the circle passing through the remaining three is zero. One of these quantities must be negative, so that one of the four points must lie within the corresponding circle.

Also by (31) the sum of the powers of these points with respect to any other circle, divided respectively by their powers with respect to the orthogonal system, is equal to unity. And by (33) the sum of the quotients of the perpendiculars from each point on any straight line, divided by the power of that point with respect to the circle which does not pass through it, is zero.

29. There is another special system of circles which is closely allied to the orthogonal

system discussed in § 25; which is of some importance.   Thus, let (1, 2) be any two circles cutting orthogonally, and let (3, 4) be their two points of intersection; the equation

$$\Pi\begin{pmatrix} x, 1, 2, 3, 4 \\ y, 1, 2, 3, 4 \end{pmatrix} = 0$$

becomes

$$\begin{vmatrix} \pi_{x,y}, & \pi_{x,1}, & \pi_{x,2}, & \pi_{x,3}, & \pi_{x,4} \\ \pi_{y,1}, & \pi_{1,1}, & 0, & 0, & 0 \\ \pi_{y,2}, & 0, & \pi_{2,2}, & 0, & 0 \\ \pi_{y,3}, & 0, & 0, & 0, & \pi_{3,4} \\ \pi_{y,4}, & 0. & 0, & \pi_{4,3}, & 0 \end{vmatrix} = 0;$$

which may be written, if we put $\pi_{3,4} = e^2$,

$$\pi_{x,y} = \frac{\pi_{x,1}.\pi_{y,1}}{\pi_{1,1}} + \frac{\pi_{x,2}.\pi_{y,2}}{\pi_{2,2}} + \frac{\pi_{x,3}.\pi_{y,4} + \pi_{x,4}.\pi_{y,3}}{e^2}, \quad . \quad . \quad . \quad . \quad . \quad (36)$$

a particular case of which is

$$0 = \frac{1}{r_1^2} + \frac{1}{r_2^2} - \frac{4}{e^2}.$$

Circles touching one another.—§§ 30–34.

30. Two circles may be said to touch externally, or internally, according as their angle of intersection $= 0$ or π, *i.e.*, if we denote the circles by x, y, then they touch externally if $\pi_{xy} = +(\pi_{x,x}.\pi_{y,y})^{\frac{1}{2}}$, and internally if $\pi_{xy} = -(\pi_{x,x}.\pi_{y,y})^{\frac{1}{2}}$.

If the four circles (1, 2, 3, 4) touch externally, we have at once, from the equation

$$\Pi\begin{pmatrix} \theta, 1, 2, 3, 4 \\ \theta, 1, 2, 3, 4 \end{pmatrix} = 0;$$

$$\begin{vmatrix} 0, & \dfrac{1}{r_1}, & \dfrac{1}{r_2}, & \dfrac{1}{r_3}, & \dfrac{1}{r_4} \\ \dfrac{1}{r_1}, & -1, & 1, & 1, & 1 \\ \dfrac{1}{r_2}, & 1, & -1, & 1, & 1 \\ \dfrac{1}{r_3}, & 1, & 1, & -1, & 1 \\ \dfrac{1}{r_4}, & 1, & 1, & 1, & -1 \end{vmatrix} = 0.$$

3 s

Whence

$$\frac{1}{r_1{}^2}+\frac{1}{r_2{}^2}+\frac{1}{r_3{}^2}+\frac{1}{r_4{}^2}=2\left(\frac{1}{r_1r_2}+\frac{1}{r_1r_3}+\frac{1}{r_1r_4}+\frac{1}{r_2r_3}+\frac{1}{r_3r_4}+\frac{1}{r_3r_4}\right); \quad . \quad . \quad . \quad (37)$$

or if r_1, r_2, r_3 be known,

$$\frac{1}{r_4}=\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\pm2\left(\frac{1}{r_2r_3}+\frac{1}{r_3r_1}+\frac{1}{r_1r_2}\right)^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad (38)$$

31. This formula is given by STEINER ('CRELLE, Journ. Math.,' vol. 1, 1826), in a paper, in which several interesting cases of series of circles touching one another are discussed : two cases may be noticed here.

Let two circles (1, 2), radii a, c, be described touching each other externally and touching another circle, radius r, internally, whose centre lies on the common diameter of the other two. Now let a circle S_1 be described touching the two former externally and the latter internally, and let a series of circles S_2, S_3, &c., be described, all touching (1, 2) externally, as well as the preceding one in the series. Let their radii be r_1, r_2, &c.

Clearly r_{n-1}, r_{n+1} are the roots of the equation

$$\frac{1}{r_n{}^2}+\frac{1}{\rho^2}+\frac{1}{a^2}+\frac{1}{c^2}=\frac{2}{\rho}\left(\frac{1}{a}+\frac{1}{c}+\frac{1}{r_n}\right)+\frac{2}{ac}+\frac{2}{r_na}+\frac{2}{r_nc},$$

therefore

$$\frac{1}{r_{n-1}}-\frac{2}{r_n}+\frac{1}{r_{n+1}}=2\left(\frac{1}{a}+\frac{1}{c}\right)=\frac{2r}{ac}.$$

And we easily find that

$$r_n=\frac{acr}{n^2r^2-ac}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

If, however, S'_1 be drawn touching (1, 2) externally, and also the line joining their centres, we shall have, if S'_n be the nth circle of this second series

$$r'_n=\frac{4acr}{(2n-1)^2r^2-4ac}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

32. If the system of circles (1, 2, 3, 4) have a common tangent circle x, say, the equation

$$\Pi\binom{x,\ 1,\ 2,\ 3,\ 4}{x,\ 1,\ 2,\ 3,\ 4}=0$$

may be written

$$\begin{vmatrix} 1, & \sqrt{\pi_{1,1}}, & \sqrt{\pi_{2,2}}, & \sqrt{\pi_{3,3}}, & \sqrt{\pi_{4,4}} \\ \sqrt{\pi_{1,1}}, & \pi_{1,1}, & \pi_{1,2}, & \pi_{1,3}, & \pi_{1,4} \\ \sqrt{\pi_{2,2}}, & \pi_{2,1}, & \pi_{2,2}, & \pi_{2,3}, & \pi_{2,4} \\ \sqrt{\pi_{3,3}}, & \pi_{3,1}, & \pi_{3,2}, & \pi_{3,3}, & \pi_{3,4} \\ \sqrt{\pi_{4,4}}, & \pi_{4,1}, & \pi_{4,2}, & \pi_{4,3}, & \pi_{4,4} \end{vmatrix}=0 ; \quad . \quad . \quad . \quad (41)$$

where the positive sign is to be taken with the radicals for external contact, and the negative for internal contact.

Now if $t_{r,s}$ denote the direct common tangent of the circles (r, s) we have

$$t^2_{r,s} = \pi_{r,s} + \sqrt{\pi_{r,r} \cdot \pi_{s,s}} \; ;$$

and if $t'_{r,s}$ denote the transverse common tangent, then

$$t'^2_{r,s} = \pi_{r,s} - \sqrt{\pi_{r,r} \cdot \pi_{s,s}}.$$

We can then deduce at once from (41)

$$\begin{vmatrix} 0, & t_{1,2}^{\;2}, & t_{1,3}^{\;2}, & t_{1,4}^{\;2} \\ t_{2,1}^{\;2}, & 0, & t_{2,3}^{\;2}, & t_{2,4}^{\;2} \\ t_{3,1}^{\;2}, & t_{3,2}^{\;2}, & 0, & t_{3,4}^{\;2} \\ t_{4,1}^{\;2}, & t_{4,2}^{\;2}, & t_{4,3}^{\;2}, & 0 \end{vmatrix} = 0,$$

or

$$t_{1,2} \cdot t_{3,4} \pm t_{1,3} \cdot t_{4,2} \pm t_{1,4} \cdot t_{2,3} = 0 \; ;$$

which is Dr. Casey's well-known formula.[*]

33. When the condition in the last article is satisfied, we can find the radius of the tangent circle by equation (31).

Thus, let the system $(5, 6, 7, 8)$ be the system orthogonal to the system $(1, 2, 3, 4)$, then we have

$$\frac{\pi_{x,1}}{\pi_{1,5}} + \frac{\pi_{x,2}}{\pi_{2,6}} + \frac{\pi_{x,3}}{\pi_{3,7}} + \frac{\pi_{x,4}}{\pi_{4,8}} = 1.$$

Thus if (x) touch each of the circles externally, we have

$$\frac{1}{r_x} = \frac{2r_1}{\pi_{1,5}} + \frac{2r_2}{\pi_{2,6}} + \frac{2r_3}{\pi_{3,7}} + \frac{2r_4}{\pi_{4,8}}. \qquad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (42)$$

34. If the system $(1, 2, 3, 4)$ be such that four other circles $(5, 6, 7, 8)$ can be drawn to touch them all, symmetrically : say each of the latter touches three of the former externally and one internally : then the equation

$$\Pi\begin{pmatrix} x, & 1, & 2, & 3, & 4 \\ y, & 5, & 6, & 7, & 8 \end{pmatrix} = 0$$

becomes

$$-8\pi_{x,y} = 2\Sigma \frac{\pi_{y,1} \cdot \pi_{x,5}}{r_1 r_5} - \left(\Sigma \frac{\pi_{y,1}}{r_1}\right)\left(\Sigma \frac{\pi_{x,5}}{r_5}\right) \; ;$$

[* If the circle (x) touches the circles $(1, 2)$ in opposite senses, then $t_{1,2}$ must be replaced by $t'_{1,2}$ in this formula.—October, 1886.]

whence as a particular case

$$\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}\right)\left(\frac{1}{r_5}+\frac{1}{r_6}+\frac{1}{r_7}+\frac{1}{r_8}\right)=2\left(\frac{1}{r_1r_5}+\frac{1}{r_2r_6}+\frac{1}{r_3r_7}+\frac{1}{r_4r_8}\right). \quad . \quad . \quad . \quad (43)$$

CHAPTER IV.—CIRCLES CONNECTED WITH A TRIANGLE.

The properties of several of these circles have been discussed in detail by the author in a paper published in the ' Quarterly Journal of Mathematics' (vol. 21, 1885). It is only proposed to discuss here some of the more general cases, which can be deduced at once from the general equation in § 6, and which by this method admit of immediate extension. The first case considered, viz., the circles which cut three given circles at given angles, is discussed by DARBOUX (' Annales de l'École Normale,' vol. 1, 1872). By triangle is meant the general case of a triangle formed by three circles.

Circles cutting Three Given Circles at Given Angles.—§§ 35–38.

35. Let (1, 2, 3) denote any given system of three circles, which cut at angles α, β, γ. If then S be any circle which cuts them at angles θ, ϕ, ψ, we have at once by § 7—denoting the radius by ρ—

$$\begin{vmatrix} 0, & \dfrac{1}{\rho}, & \dfrac{1}{r_1}, & \dfrac{1}{r_2}, & \dfrac{1}{r_3} \\[2mm] \dfrac{1}{\rho}, & -1, & \cos\theta, & \cos\phi, & \cos\psi \\[2mm] \dfrac{1}{r_1}, & \cos\theta, & -1, & \cos\gamma, & \cos\beta \\[2mm] \dfrac{1}{r_2}, & \cos\phi, & \cos\gamma, & -1, & \cos\alpha \\[2mm] \dfrac{1}{r_3}, & \cos\psi, & \cos\beta, & \cos\alpha, & -1 \end{vmatrix} = 0. \quad . \quad . \quad . \quad . \quad (44)$$

Thus we obtain a quadratic for ρ ; and two circles can in general be drawn cutting the given circles at the given angles.

Now let either of these circles cut the orthogonal circle of the system (1, 2, 3), at the angle ω—denoting the orthogonal circle by the symbol 4, and its radius by r ; the equation

$$\Pi\begin{pmatrix} S, 4, 1, 2, 3 \\ S, 4, 1, 2, 3 \end{pmatrix}=0$$

becomes

$$\begin{vmatrix} -1, & \cos\omega, & \cos\theta, & \cos\phi, & \cos\psi \\ \cos\omega, & -1, & 0, & 0, & 0 \\ \cos\theta, & 0, & -1, & \cos\gamma, & \cos\beta \\ \cos\phi, & 0, & \cos\gamma, & -1, & \cos\alpha \\ \cos\psi, & 0, & \cos\beta, & \cos\alpha, & -1 \end{vmatrix} = 0 \; ;$$

or

$$\sin^2\omega \begin{vmatrix} -1, & \cos\gamma, & \cos\beta \\ \cos\gamma, & -1, & \cos\alpha \\ \cos\beta, & \cos\alpha, & -1 \end{vmatrix} = \begin{vmatrix} 0, & \cos\theta, & \cos\phi, & \cos\psi \\ \cos\theta, & -1, & \cos\gamma, & \cos\beta \\ \cos\phi, & \cos\gamma, & -1, & \cos\alpha \\ \cos\psi, & \cos\beta, & \cos\alpha, & -1 \end{vmatrix} \; . \quad (45)$$

It thus appears that each of the circles, which can be drawn to cut the system (1, 2, 3) at angles θ, ϕ, ψ, cuts the orthogonal circle at one of the angles ω or $\pi-\omega$. It is otherwise evident that these two circles are such that one is the inverse of the other with respect to the circle which cuts the system (1, 2, 3) orthogonally.

Let us denote the two circles by S, S′, and their radii by ρ, ρ'. We have by

$$\Pi \begin{pmatrix} x, & 4, & 1, & 2, & 3 \\ \theta, & 4, & 1, & 2, & 3 \end{pmatrix} = 0,$$

the equation

$$\begin{vmatrix} \dfrac{1}{\rho}, & \dfrac{1}{r}, & \dfrac{1}{r_1}, & \dfrac{1}{r_2}, & \dfrac{1}{r_3}, \\ \cos\omega, & -1, & 0, & 0, & 0 \\ \cos\theta, & 0, & -1, & \cos\gamma, & \cos\beta \\ \cos\phi, & 0, & \cos\gamma, & -1, & \cos\alpha \\ \cos\psi, & 0, & \cos\beta, & \cos\alpha, & -1 \end{vmatrix} = 0 \; ; \quad . \quad . \quad . \quad (46)$$

and we also get a similar equation for ρ'.

It appears, then, that the circle S′, which is the inverse of S, cuts the orthogonal circle at the angle $\pi-\omega$. It may happen, however, that the roots of equation (44) are of opposite sign; in this case, the circle S′ evidently cuts the given system (1, 2, 3) at angles $\pi-\theta$, $\pi-\phi$, $\pi-\psi$, and the circle orthogonal to these at the angle ω.

ρ and ρ' being the roots of (44), we have at once by (46)

$$\frac{1}{\rho} - \frac{1}{\rho'} + \frac{2\cos\omega}{r} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

i.e.,

$$\frac{1}{\pi_{\mathrm{S},4}} + \frac{1}{\pi_{\mathrm{S}',4}} = -\frac{1}{r^2} = \frac{2}{\pi_{4,4}}.$$

Hence the two circles S, S′ are real, coincident, or imaginary, according as $\frac{\cos^2 \omega}{r^2}$ is positive, zero, or negative.

But by § 18 the sign of r^2 is the same as the sign of

$$\Pi\begin{pmatrix} 1, 2, 3 \\ 1, 2, 3 \end{pmatrix} \quad \text{or} \quad \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma + 2\cos \alpha \cos \beta \cos \gamma - 1.$$

Hence, by (45), S, S′ are real, coincident, or imaginary, according as

$$\begin{vmatrix} -1, & \cos\theta, & \cos\phi, & \cos\psi \\ \cos\theta, & -1, & \cos\gamma, & \cos\beta \\ \cos\phi, & \cos\gamma, & -1, & \cos\alpha \\ \cos\psi, & \cos\beta, & \cos\alpha, & -1 \end{vmatrix}$$

is negative, zero, or positive, *i.e.*, according as

$$\Pi\begin{pmatrix} S, 1, 2, 3 \\ S, 1, 2, 3 \end{pmatrix}$$

is negative, zero, or positive.

36. It is clear that four pairs of circles can be drawn to cut the given system (1, 2, 3) at angles equal to θ, ϕ, ψ, or their supplements. The radii of these eight circles are connected by a remarkable relation. Let them be denoted by ρ, ρ'; ρ_1, ρ'_1; ρ_2, ρ'_2; ρ_3, ρ'_3. By equation (46) we have

$$\begin{vmatrix} \frac{1}{\rho}+\frac{1}{\rho'}, & \frac{1}{r_1}, & \frac{1}{r_2}, & \frac{1}{r_3} \\ \cos\theta, & -1, & \cos\gamma, & \cos\beta \\ \cos\phi, & \cos\gamma, & -1, & \cos\alpha \\ \cos\psi, & \cos\beta, & \cos\alpha, & -1 \end{vmatrix} = 0,$$

or

$$\frac{1}{\rho}+\frac{1}{\rho'}=\text{F}\cos\theta+\text{G}\cos\phi+\text{H}\cos\psi ; \quad \ldots \ldots \quad (48)$$

similarly, we shall have,

$$\frac{1}{\rho_1}+\frac{1}{\rho'_1}=-\text{F}\cos\theta+\text{G}\cos\phi+\text{H}\cos\psi,$$

$$\frac{1}{\rho_2}+\frac{1}{\rho'_2}=\ \ \text{F}\cos\theta-\text{G}\cos\phi+\text{H}\cos\psi,$$

$$\frac{1}{\rho_3}+\frac{1}{\rho'_3}=\ \ \text{F}\cos\theta+\text{G}\cos\phi-\text{H}\cos\psi.$$

Hence

$$\frac{1}{\rho}+\frac{1}{\rho'}=\frac{1}{\rho_1}+\frac{1}{\rho'_1}+\frac{1}{\rho_2}+\frac{1}{\rho'_2}+\frac{1}{\rho_3}+\frac{1}{\rho'_3}. \quad \ldots \ldots \quad (49)$$

37. We also obtain some interesting results by considering another group of circles allied to the pair considered in § 35. Thus let $2s=\theta+\phi+\psi$; let ρ_1, ρ'_1 be the radii of the pair of circles cutting the system (1, 2, 3) at angles $2s$, ψ, ϕ; let ρ_2, ρ'_2 be the radii of the pair cutting the system at angles ψ, $2s$, θ; and ρ_3, ρ'_3 the radii of the pair cutting the system at angles ϕ, θ, $2s$.

As in (48), we have

$$\frac{1}{\rho}+\frac{1}{\rho'}=\text{F}\cos\theta+\text{G}\cos\phi+\text{H}\cos\psi,$$

$$\frac{1}{\rho_1}+\frac{1}{\rho'_1}=\text{F}\cos 2s+\text{G}\cos\psi+\text{H}\cos\phi,$$

$$\frac{1}{\rho_2}+\frac{1}{\rho'_2}=\text{F}\cos\psi+\text{G}\cos 2s+\text{H}\cos\theta,$$

$$\frac{1}{\rho_3}+\frac{1}{\rho'_3}=\text{F}\cos\phi+\text{G}\cos\theta+\text{H}\cos 2s.$$

Hence, by addition,

$$\frac{1}{\rho}+\frac{1}{\rho'}+\frac{1}{\rho_1}+\frac{1}{\rho'_1}+\frac{1}{\rho_2}+\frac{1}{\rho'_2}+\frac{1}{\rho_3}+\frac{1}{\rho'_3}$$

$$=(\text{F}+\text{G}+\text{H})(\cos\theta+\cos\phi+\cos\psi+\cos 2s)$$

$$=\left(\frac{1}{\text{R}}+\frac{1}{\text{R}'}\right)4\cos\tfrac{1}{2}(\phi+\psi).\cos\tfrac{1}{2}(\psi+\theta).\cos\tfrac{1}{2}(\theta+\phi)\;;\quad .\quad .\quad .\quad .\quad (50)$$

where R, R' are the radii of the circles which touch the given circles externally.

Similarly

$$\frac{1}{\rho}+\frac{1}{\rho'}+\frac{1}{\rho_1}+\frac{1}{\rho'_1}-\frac{1}{\rho_2}-\frac{1}{\rho'_2}-\frac{1}{\rho_3}-\frac{1}{\rho'_3}$$

$$=(\text{F}-\text{G}-\text{H})(\cos\theta+\cos 2s-\cos\phi-\cos\psi)$$

$$=\left(\frac{1}{\text{R}_1}+\frac{1}{\text{R}'_1}\right)4\cos\tfrac{1}{2}(\phi+\psi).\sin\tfrac{1}{2}(\psi+\theta).\sin\tfrac{1}{2}(\theta+\phi)\;;$$

where R_1, R'_1 are the radii of the circles which touch the circle (1) internally and the circles (2, 3) externally.

38. The problem of drawing a circle to cut four given circles at equal angles has been discussed by DARBOUX, in his paper cited above, who makes the solution depend on that of drawing a circle to cut three given circles at given angles. Given four circles, say (1, 2, 3, 4), we can easily find the angle at which a circle can cut them, and the radius of the circle. For let this angle be ϕ, then, if we denote the circle by S, we have

$$\Pi\begin{pmatrix}\text{S, 1, 2, }\overset{*}{3}\text{, 4}\\ \text{S, 1, 2, 3, 4}\end{pmatrix}=0,$$

whence

$$
\begin{vmatrix}
-1, & \cos\phi, & \cos\phi, & \cos\phi, & \cos\phi \\
\cos\phi, & -1, & \cos\omega_{1,2}, & \cos\omega_{1,3}, & \cos\omega_{1,4} \\
\cos\phi, & \cos\omega_{2,1}, & -1, & \cos\omega_{2,3}, & \cos\omega_{2,4} \\
\cos\phi, & \cos\omega_{3,1}, & \cos\omega_{3,2}, & -1, & \cos\omega_{3,4} \\
\cos\phi, & \cos\omega_{4,1}, & \cos\omega_{4,2}, & \cos\omega_{4,3}, & -1
\end{vmatrix} = 0 ;
$$

where $\omega_{1,2}$, $\omega_{1,3}$, &c., are the angles of intersection of the system $(1, 2, 3, 4)$.

We have then to determine ϕ, the equation

$$
\begin{vmatrix}
-\sec^2\phi, & 1, & 1, & 1, & 1, \\
1, & -1, & \cos\omega_{1,2}, & \cos\omega_{1,3}, & \cos\omega_{1,4} \\
1, & \cos\omega_{2,1}, & -1, & \cos\omega_{2,3}, & \cos\omega_{2,4} \\
1, & \cos\omega_{3,1}, & \cos\omega_{3,2}, & -1, & \cos\omega_{3,4} \\
1, & \cos\omega_{4,1}, & \cos\omega_{4,2}, & \cos\omega_{4,3}, & -1
\end{vmatrix} = 0 . \quad . \quad . \quad (51)
$$

Also since

$$
\Pi\begin{pmatrix} S, 1, 2, 3, 4 \\ \theta, 1, 2, 3, 4 \end{pmatrix} = 0 ;
$$

we obtain at once, if ρ be the radius of the circle,

$$
\begin{vmatrix}
\dfrac{\sec\phi}{\rho}, & \dfrac{1}{r_1}, & \dfrac{1}{r_2}, & \dfrac{1}{r_3}, & \dfrac{1}{r_4} \\
1, & -1, & \cos\omega_{1,2}, & \cos\omega_{1,3}, & \cos\omega_{1,4} \\
1, & \cos\omega_{2,1}, & -1, & \cos\omega_{2,3}, & \cos\omega_{2,4} \\
1, & \cos\omega_{3,1}, & \cos\omega_{3,2}, & -1, & \cos\omega_{3,4} \\
1, & \cos\omega_{4,1}, & \cos\omega_{4,2}, & \cos\omega_{4,3}, & -1
\end{vmatrix} = 0 . \quad . \quad . \quad (52)
$$

We thus see that only one circle can be drawn ; (51) determines the angle at which this circle cuts the given system.

For instance, if the four given circles cut orthogonally, we have

$$
\sec\phi = 2,
$$

$$
\frac{2}{\rho} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4}.
$$

This circle will be imaginary, since one of the four $(1, 2, 3, 4)$ is so.

The Circles which pass through three of the points of intersection of three Circles.
§§ 39–41.

39. Let the given systems of circles be denoted by (1, 2, 3), and their orthogonal circle by the symbol (4). Let P, Q, R, P′, Q′, R′ be the six points of intersection of the three circles, the points P, Q, R being situated within the triangle formed by the centres of the circles. Let S be the circle which passes through the points P, Q, R.

We have, then,

$$\Pi\binom{S,\ 2,\ 3}{S,\ 2,\ 3}=\Pi\binom{S,\ 3,\ 1}{S,\ 3,\ 1}=\Pi\binom{S,\ 1,\ 2}{S,\ 1,\ 2}=0.$$

Hence we have by a theorem of determinants

and

$$\left.\begin{aligned}\left\{\Pi\binom{S,\ 1,\ 3}{S,\ 2,\ 3}\right\}^{2}&=-\Pi\binom{S,\ 3}{S,\ 3}\times\Pi\binom{S,\ 1,\ 2,\ 3}{S,\ 1,\ 2,\ 3}\\[2mm]\left\{\Pi\binom{S,\ 2,\ 3}{1,\ 2,\ 3}\right\}^{2}&=-\Pi\binom{2,\ 3}{2,\ 3}\times\Pi\binom{S,\ 1,\ 2,\ 3}{S,\ 1,\ 2,\ 3}\end{aligned}\right\}\quad\cdots\cdots\quad(53)$$

But since

$$\Pi\binom{S,\ 1,\ 2,\ 3,\ \theta}{S,\ 1,\ 2,\ 3,\ 4}=0,$$

we have

$$\pi_{S,4}.\Pi\binom{1,\ 2,\ 3,\ \theta}{1,\ 2,\ 3,\ S}-\Pi\binom{S,\ 1,\ 2,\ 3}{S,\ 1,\ 2,\ 3}=0\ ;$$

or

$$\Pi\binom{S,\ 1,\ 2,\ 3}{S,\ 1,\ 2,\ 3}=\pi_{S,4}\left\{\Pi\binom{1,\ 2,\ 3}{1,\ 2,\ 3}-\Pi\binom{S,\ 2,\ 3}{1,\ 2,\ 3}-\Pi\binom{S,\ 3,\ 1}{1,\ 2,\ 3}-\Pi\binom{S,\ 1,\ 2}{1,\ 2,\ 3}\right\},\quad\cdots\quad(54)$$

and since

$$\Pi\binom{S,\ 1,\ 2,\ 3,\ 4}{S,\ 1,\ 2,\ 3,\ 4}=0,$$

$$\pi_{4,4}.\Pi\binom{S,\ 1,\ 2,\ 3}{S,\ 1,\ 2,\ 3}=\pi^{2}{}_{S,4}.\Pi\binom{1,\ 2,\ 3}{1,\ 2,\ 3}.\quad\cdots\cdots\cdots\quad(55)$$

But by § 18,

$$\Pi\binom{1,\ 2,\ 3}{1,\ 2,\ 3}=-16\pi_{4,4}\left\{\Delta(1,\ 2,\ 3)\right\}^{2},$$

hence by means of (53), (54) becomes

$$\frac{\pi_{4,4}-\pi_{4,S}}{\pi_{4,S}}.4\Delta(1,\ 2,\ 3)=\left\{\Pi\binom{2,\ 3}{2,\ 3}\right\}^{\frac{1}{2}}+\left\{\Pi\binom{3,\ 1}{3,\ 1}\right\}^{\frac{1}{2}}+\left\{\Pi\binom{1,\ 2}{1,\ 2}\right\}^{\frac{1}{2}}.$$

But

$$\Pi\binom{2,\ 3}{2,\ 3}=16\{\Delta(P,\ 2,\ 3)\}^{2}.$$

3 T

Thus we may write this equation

$$\frac{\pi_{4,4} - \pi_{4,S}}{\pi_{4,S}} = \frac{\Delta(P, 2, 3) + \Delta(Q, 3, 1) + \Delta(R, 1, 2)}{\Delta(1, 2, 3)} \, ;$$

or

$$\frac{\rho \cos \omega}{r} = \frac{\Delta(1, 2, 3)}{\Delta(1, 2, 3) + \Delta(P, 2, 3) + \Delta(Q, 3, 1) + \Delta(R, 1, 2)}, \quad \cdot \quad \cdot \quad \cdot \quad (56)$$

where ω is the angle of intersection of S and the orthogonal circle of the system
(1, 2, 3), and ρ, r are the radii of the circles (S, 4).

This formula is easily adapted for the circles (PQR'), &c., by taking the area of the
triangle (P', 2, 3) as of opposite sign to $\Delta(P, 2, 3)$.

Again, if $\kappa_{S,S}$, $\kappa_{S,1}$, &c. denote the minors of $\pi_{S,S}$, $\pi_{S,1}$, &c. in $\Pi\begin{pmatrix} S, 1, 2, 3 \\ S, 1, 2, 3 \end{pmatrix}$; and if $\mu_{1,1}$,

$\mu_{1,2}$, &c. denote the minors of $\pi_{1,1}$, $\pi_{1,2}$, &c. in $\Pi\begin{pmatrix} 1, 2, 3 \\ 1, 2, 3 \end{pmatrix}$, we shall have from (55)

$$\frac{\pi_{4,S}^2}{\pi_{4,4}} . \Pi\begin{pmatrix} 1, 2, 3 \\ 1, 2, 3 \end{pmatrix} \times \mu\begin{pmatrix} 1, 2, 3 \\ 1, 2, 3 \end{pmatrix} = \Pi\begin{pmatrix} S, 1, 2, 3 \\ S, 1, 2, 3 \end{pmatrix} \times \mu\begin{pmatrix} 1, 2, 3 \\ 1, 2, 3 \end{pmatrix}$$

$$= \begin{vmatrix} \pi_{S,S}.\kappa_{S,S}, & \kappa_{S,1}, & \kappa_{S,2}, & \kappa_{S,3} \\ \kappa_{1,S}, & \mu_{1,1}, & \mu_{1,2}, & \mu_{1,3} \\ \kappa_{2,S}, & \mu_{2,1}, & \mu_{2,2}, & \mu_{2,3} \\ \kappa_{3,S}, & \mu_{3,1}, & \mu_{3,2}, & \mu_{3,3} \end{vmatrix} .$$

But by (53)

$$\kappa_{S,1}^2 = \left\{ \Pi\begin{pmatrix} S, 2, 3 \\ 1, 2, 3 \end{pmatrix} \right\}^2 = -\Pi\begin{pmatrix} 2, 3 \\ 2, 3 \end{pmatrix} . \Pi\begin{pmatrix} S, 1, 2, 3 \\ S, 1, 2, 3 \end{pmatrix} = -\mu_{1,1} . \Pi\begin{pmatrix} S, 1, 2, 3 \\ S, 1, 2, 3 \end{pmatrix}.$$

Therefore

$$\frac{\pi_{S,S}.\pi_{4,4} - \pi_{4,S}^2}{\pi_{4,S}^2} \times \left\{ \Pi\begin{pmatrix} 1, 2, 3 \\ 1, 2, 3 \end{pmatrix} \right\}^2 = \begin{vmatrix} 0, & \sqrt{\mu_{1,1}}, & \sqrt{\mu_{2,2}}, & \sqrt{\mu_{3,3}} \\ \sqrt{\mu_{1,1}}, & \mu_{1,1}, & \mu_{1,2}, & \mu_{1,3} \\ \sqrt{\mu_{2,2}}, & \mu_{2,1}, & \mu_{2,2}, & \mu_{2,3} \\ \sqrt{\mu_{3,3}}, & \mu_{3,1}, & \mu_{3,2}, & \mu_{3,3} \end{vmatrix} . \quad \cdot \quad \cdot \quad (57)$$

But

$$\frac{\pi_{S,S}\pi_{4,4} - \pi_{4,S}^2}{\pi_{4,S}^2} = \tan^2 \omega \, ;$$

and

$$\mu_{1,1} = 16\{\Delta(P, 2, 3)\}^2.$$

So that we may write the above equation

$$64\,r^4\tan^2\omega\{\Delta(1,2,3)\}^4=\begin{vmatrix} 0, & \Delta(P,2,3), & \Delta(Q,3,1), & \Delta(R,1,2) \\ \Delta(P,2,3), & \mu_{1,1}, & \mu_{1,2}, & \mu_{1,3} \\ \Delta(Q,3,1), & \mu_{2,1}, & \mu_{2,2}, & \mu_{2,3} \\ \Delta(R,1,2), & \mu_{3,1}, & \mu_{3,2}, & \mu_{3,3} \end{vmatrix}\;.\;(58)$$

Similarly we can find ω for the circles (P', Q, R), &c.

40. If the given system $(1, 2, 3)$ intersect at angles α, β, γ, equation (57) reduces easily to

$$\cos^2\omega=\sec s\,.\,\cos(s-\alpha)\,.\,\cos(s-\beta)\,.\,\cos(s-\gamma),\quad\ldots\ldots(59)$$

where

$$2s=\alpha+\beta+\gamma.$$

Also for circle $(P'QR)$ we shall find

$$\cos^2\omega=\cos s\,.\,\sec(s-\alpha)\,.\,\cos(s-\beta)\,.\,\cos(s-\gamma).$$

41. It may be noticed that the system $(P, Q, R, 4)$ is the orthogonal system to $(1, 2, 3, S)$—(see § 27).

We have at once from equation (35)

$$\frac{1}{\pi_{P,1}}+\frac{1}{\pi_{Q,2}}+\frac{1}{\pi_{R,3}}+\frac{1}{\pi_{S,4}}=0.\;\ldots\ldots(60)$$

If S' be the circle through (P', Q', R'), we have

$$\frac{1}{\pi_{P',1}}+\frac{1}{\pi_{Q',2}}+\frac{1}{\pi_{R',3}}+\frac{1}{\pi_{S',4}}=0.$$

But by (47), since S' is the inverse of S with respect to the circle (4),

$$\frac{1}{\pi_{S,4}}+\frac{1}{\pi_{S',4}}=\frac{2}{\pi_{4,4}}.$$

Hence we have

$$\frac{1}{\pi_{P,1}}+\frac{1}{\pi_{P',1}}+\frac{1}{\pi_{Q,2}}+\frac{1}{\pi_{Q',2}}+\frac{1}{\pi_{R,3}}+\frac{1}{\pi_{R',3}}+\frac{2}{\pi_{4,4}}=0;\;\ldots\ldots(61)$$

or the sum of the reciprocals of the squares of the tangents, from the points of intersection of three circles to the circles, is equal to the reciprocal of the square of the radius of the circle which cuts the circles orthogonally.

Again, from equation (31), we shall have

$$\frac{\pi_{S,1}}{\pi_{P,1}} + \frac{\pi_{S,2}}{\pi_{Q,2}} + \frac{\pi_{S,3}}{\pi_{R,3}} + \frac{\pi_{S,S}}{\pi_{S,4}} = 1, \quad \ldots \ldots \ldots \quad (62)$$

or, if S cut the system (1, 2, 3) at angles ϕ_1, ϕ_2, ϕ_3, this may be written

$$\frac{1}{\rho} = \frac{2r_1 \cos \phi_1}{\pi_{P,1}} + \frac{2r_2 \cos \phi_2}{\pi_{Q,2}} + \frac{2r_3 \cos \phi_3}{\pi_{R,3}} - \frac{1}{r \cos \omega}. \quad \ldots \ldots \quad (63)$$

This with equation (60) is sufficient to determine ω and ρ. Also by drawing a figure we shall see at once that

$$\phi_2 + \phi_3 + \alpha = \phi_3 + \phi_1 + \beta = \phi_1 + \phi_2 + \gamma = \pi.$$

So that, if $2s = \alpha + \beta + \gamma$,

$$\phi_1 = \tfrac{1}{2}\pi - (s - \alpha); \quad \phi_2 = \tfrac{1}{2}\pi - (s - \beta); \quad \phi_3 = \tfrac{1}{2}\pi - (s - \gamma).$$

The Circles which touch three given Circles.—§§ 42–46.

42. Let (1, 2, 3) be the given system, and let (4) denote the orthogonal circle of the system: then if S be the circle which touches all the circles externally, and ω its angle of intersection with (4), we shall have, since,

$$\pi^2_{S,1} = \pi_{1,1}.\pi_{S,S}; \quad \&c.,$$

and

$$\pi_{4,4}.\Pi\begin{pmatrix} S, 1, 2, 3 \\ S, 1, 2, 3 \end{pmatrix} = \pi^2_{4,S}.\Pi\begin{pmatrix} 1, 2, 3 \\ 1, 2, 3 \end{pmatrix};$$

$$(\cos^2 \omega - 1).\Pi\begin{pmatrix} 1, 2, 3 \\ 1, 2, 3 \end{pmatrix} = \begin{vmatrix} 0, & \sqrt{\pi_{1,1}}, & \sqrt{\pi_{2,2}}, & \sqrt{\pi_{3,3}} \\ \sqrt{\pi_{1,1}}, & \pi_{1,1}, & \pi_{1,2}, & \pi_{1,3} \\ \sqrt{\pi_{2,2}}, & \pi_{2,1}, & \pi_{2,2}, & \pi_{2,3} \\ \sqrt{\pi_{3,3}}, & \pi_{3,1}, & \pi_{3,2}, & \pi_{3,3} \end{vmatrix} \quad \ldots \quad (64)$$

By giving the expressions $\sqrt{\pi_{1,1}}$, &c., different signs, we obtain the values of $\cos \omega$ for the other pairs of tangent circles; and it is clear that there are four pairs of such circles.

43. If α, β, γ be the angles of intersection of the system (1, 2, 3), and ω, ω_1, ω_2, ω_3 the angles of intersection of the four pairs of tangent circles with the orthogonal circle of the system, we can easily deduce from (64) the formulæ

$$\left.\begin{array}{l} K \cos^2 \omega = 2(1 + \cos \alpha)(1 + \cos \beta)(1 + \cos \gamma) \\ K \cos^2 \omega_1 = 2(1 + \cos \alpha)(1 - \cos \beta)(1 - \cos \gamma) \\ K \cos^2 \omega_2 = 2(1 - \cos \alpha)(1 + \cos \beta)(1 - \cos \gamma) \\ K \cos^2 \omega_3 = 2(1 - \cos \alpha)(1 - \cos \beta)(1 + \cos \gamma) \end{array}\right\}, \quad \cdots \cdots \quad (65)$$

where

$$K = \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma + 2 \cos \alpha \cos \beta \cos \gamma - 1.$$

44. The radii of the circles are given at once by the formula

$$\Pi\binom{S, 1, 2, 3, 4}{\theta, 1, 2, 3, 4} = 0.$$

Thus, if ρ be the radius of circle touching the system $(1, 2, 3)$ externally, we have

$$\begin{vmatrix} \dfrac{1}{\rho}, & \dfrac{1}{r_1}, & \dfrac{1}{r_2}, & \dfrac{1}{r_3}, & \dfrac{1}{r} \\ 1, & -1, & \cos \gamma, & \cos \beta, & 0 \\ 1, & \cos \gamma, & -1, & \cos \alpha, & 0 \\ 1, & \cos \beta, & \cos \alpha, & -1, & 0 \\ \cos \omega, & 0, & 0, & 0, & -1 \end{vmatrix} = 0,$$

or,

$$\begin{vmatrix} \dfrac{1}{\rho} + \dfrac{\cos \omega}{r}, & \dfrac{1}{r_1}, & \dfrac{1}{r_2}, & \dfrac{1}{r_3} \\ 1, & -1, & \cos \gamma, & \cos \beta \\ 1, & \cos \gamma, & -1, & \cos \alpha \\ 1, & \cos \beta, & \cos \alpha, & -1 \end{vmatrix} = 0. \quad \cdots \cdots \quad (66)$$

If we denote the radii of the pairs of tangent circles by (ρ, ρ') (ρ_1, ρ'_1) (ρ_2, ρ'_2) (ρ_3, ρ'_3), we have, by (49)

$$\frac{1}{\rho} + \frac{1}{\rho'} = \frac{1}{\rho_1} + \frac{1}{\rho'_1} + \frac{1}{\rho_2} + \frac{1}{\rho'_2} + \frac{1}{\rho_3} + \frac{1}{\rho'_3},$$

a theorem first given by Mr. Cox.—('Quart. Journ, Math.,' vol. 19, 1883, p. 99.)

45. Let $(5, 6, 7, 8)$ denote a system of circles formed by taking one of each pair of tangent circles of the system $(1, 2, 3)$. This can be done in sixteen ways:—We may show that eight of these sixteen groups are touched respectively by eight other circles.

Let z be the circle which touches the group $(5, 6, 7, 8)$: let z touch 5 internally and $(6, 7, 8)$ externally ; then, since

$$\Pi\binom{z, 4, 1, 2, 3}{x, 5, 6, 7, 8} = 0,$$

we have, giving x the successive values 4, 1, 2, 3 :—

$$\left.\begin{aligned}
A_1 \cos \omega_{z,4} &= 4 \\
A_1 \cos \omega_{z,1} &= A_2 - A_3 \cos \gamma - A_4 \cos \beta \\
A_1 \cos \omega_{z,2} &= -A_2 \cos \gamma + A_3 - A_4 \cos \alpha \\
A_1 \cos \omega_{z,3} &= -A_2 \cos \beta - A_3 \cos \alpha + A_4
\end{aligned}\right\}, \quad \ldots \quad (66^*)$$

where

$$\left.\begin{aligned}
A_1 &= \cos \omega_{4,5} - \cos \omega_{4,6} - \cos \omega_{4,7} - \cos \omega_{4,8} \\
A_2 &= -\cos \omega_{4,5} + \cos \omega_{4,6} - \cos \omega_{4,7} - \cos \omega_{4,8} \\
A_3 &= -\cos \omega_{4,5} - \cos \omega_{4,6} + \cos \omega_{4,7} - \cos \omega_{4,8} \\
A_4 &= -\cos \omega_{4,5} - \cos \omega_{4,6} - \cos \omega_{4,7} + \cos \omega_{4,8}
\end{aligned}\right\}.$$

But we also have

$$\Pi\begin{pmatrix} z, 4, 1, 2, 3 \\ z, 4, 1, 2, 3 \end{pmatrix} = 0.$$

Therefore

$$-A_1 + A_2 \cos \omega_{1,z} + A_3 \cos \omega_{2,z} + A_4 \cos \omega_{3,z} + 4 \cos \omega_{4,z} = 0.$$

Hence we must have

$$16 - A_1^2 + A_2^2 + A_3^2 + A_4^2 - 2A_3 A_4 \cos \alpha - 2A_4 A_2 \cos \beta - 2A_2 A_3 \cos \gamma = 0,$$

which may be written

$$16 + 2\cos^2 \omega_{4,5}(1 - \cos \alpha - \cos \beta - \cos \gamma)$$
$$+ 2\cos^2 \omega_{4,6}(1 - \cos \alpha + \cos \beta + \cos \gamma)$$
$$+ 2\cos^2 \omega_{4,7}(1 + \cos \alpha - \cos \beta + \cos \gamma)$$
$$+ 2\cos^2 \omega_{4,8}(1 + \cos \alpha + \cos \beta - \cos \gamma)$$
$$+ 2(1 - \cos \alpha) \cos \omega_{4,5}. \cos \omega_{4,6} - 2(1 + \cos \alpha) \cos \omega_{4,7}. \cos \omega_{4,8}$$
$$+ 2(1 - \cos \beta) \cos \omega_{4,5}. \cos \omega_{4,7} - 2(1 + \cos \beta) \cos \omega_{4,6}. \cos \omega_{4,8}$$
$$+ 2(1 - \cos \gamma) \cos \omega_{4,5}. \cos \omega_{4,8} - 2(1 + \cos \gamma) \cos \omega_{4,6}. \cos \omega_{4,7} = 0.$$

Referring to (65) we see that this equation is satisfied, provided we choose the groups of circles so that

$$\cos \omega_{4,5}. \cos \omega_{4,6}. \cos \omega_{4,7}. \cos \omega_{4,8} \quad \text{is positive.}$$

Thus if we denote the tangent circles of the system (1, 2, 3) by the symbols (τ, τ'), (τ_1, τ'_1), (τ_2, τ'_2), (τ_3, τ'_3), where $\tau, \tau_1, \tau_2, \tau_3$ correspond to the positive values of $\cos \omega$, $\cos \omega_1$, &c., as given by (65), then we see that the groups

$$\tau, \ \tau_1, \ \tau_2, \ \tau_3 \qquad\qquad \tau', \ \tau'_1, \ \tau'_2, \ \tau'_3$$

$$\tau, \ \tau_1, \ \tau'_2, \ \tau'_3 \qquad\qquad \tau', \ \tau'_1, \ \tau_2, \ \tau_3$$

$$\tau, \ \tau'_1, \ \tau_2, \ \tau'_3 \qquad\qquad \tau', \ \tau_1, \ \tau'_2, \ \tau_3$$

$$\tau, \ \tau'_1, \ \tau'_2, \ \tau_3 \qquad\qquad \tau', \ \tau_1, \ \tau_2, \ \tau'_3$$

are touched by circles S, S_1, S_2, S_3; S', S'_1, S'_2, S'_3. Also each of these circles, S say, touches τ internally and the others externally; and if S touch the group τ, τ_1, τ_2, τ_3, and S' the group τ', τ'_1, τ'_2, τ'_3, then S' is the inverse of S with respect to the orthogonal circle of the system (1, 2, 3). These circles are usually called Dr. HART's circles.

46. We can easily deduce from formulæ (66) that

$$\left. \begin{array}{lll} \text{S, S' cut (1, 2, 3) at angles} & \beta-\gamma, \ \gamma-\alpha, \ \alpha-\beta \\ S_1, S'_1 \ \text{,, (1, 2, 3)} \quad \text{,,} & \beta-\gamma, \ \gamma+\alpha, \ \alpha+\beta \\ S_2, S'_2 \ \text{,, (1, 2, 3)} \quad \text{,,} & \beta+\gamma, \ \gamma-\alpha, \ \alpha+\beta \\ S_3, S'_3 \ \text{,, (1, 2, 3)} \quad \text{,,} & \beta+\gamma, \ \gamma+\alpha, \ \alpha-\beta \end{array} \right\} \quad . \quad . \quad . \quad . \quad (67)$$

Also, if the angles of intersection with the orthogonal circle of (1, 2, 3) of the pairs (S, S'), &c., be ϖ, ϖ_1, ϖ_2, ϖ_3, we shall have

$$\left. \begin{array}{l} \cos^2 \varpi = 4 \sec s \cdot \cos (s-\alpha) \cdot \cos (s-\beta) \cdot \cos (s-\gamma) \\ \cos^2 \varpi_1 = 4 \cos s \cdot \sec (s-\alpha) \cdot \cos (s-\beta) \cdot \cos (s-\gamma) \\ \cos^2 \varpi_2 = 4 \cos s \cdot \cos (s-\alpha) \cdot \sec (s-\beta) \cdot \cos (s-\gamma) \\ \cos^2 \varpi_3 = 4 \cos s \cdot \cos (s-\alpha) \cdot \cos (s-\beta) \cdot \sec (s-\gamma) \end{array} \right\} .$$

Referring to § 40 we see that if the given circles (1, 2, 3) intersect in the points P, Q, R, P', Q', R', and the circle (P, Q, R) cut the orthogonal circle to (1, 2, 3) at the angle ω, then

$$\cos \varpi = 2 \cos \omega.$$

Hence, if ρ, ρ' be the radii of the circles S, S', and R, R' the radii of the circles (P, Q, R), (P', Q', R'), then

$$\frac{1}{\rho} - \frac{1}{\rho'} = 2 \left(\frac{1}{R} - \frac{1}{R'} \right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

Similarly each pair of Dr. HART's circles is connected with a corresponding pair of the circles which can be drawn through the points P, Q, R, P', Q', R' by a formula which is analogous to that which connects the radius of the nine-points circle of a plane triangle with the radius of the circum-circle.

Three circles produce, as may be seen by drawing a figure, four pairs of triangles, each pair consisting of a triangle, and its inverse with respect to the orthogonal circle. Thus, supposing α, β, γ are the angles of the triangle P, Q, R, the angles of the triangle P', Q', R' are either α, β, γ or else $\pi-\alpha$, $\pi-\beta$, $\pi-\gamma$. Again, the angles of P', Q, R are α, $\pi-\beta$, $\pi-\gamma$. Hence, having obtained the formulæ for the radius of any circle connected with a particular triangle, we can easily obtain the formulæ for the other circles. It is also evident that there must be eight circles corresponding to the circum-circle, and eight circles corresponding to the nine-points circle of a plane triangle.

CHAPTER V.—POWER-COORDINATES.

Definition.— §§ 47–50.

47. We have already seen that any circle (straight line or point) is completely determined when its powers are known with respect to any four circles which have not a common orthogonal circle. Hence, given four such circles, which may be called the system of reference, any multiples, the same or different, of the powers of a circle (straight line or point) with respect to them, may be defined as its power-coordinates.

We shall find it convenient to denote the coordinates of any circle by $\xi\eta\zeta\omega$; the coordinates of any point by $xyzw$; and the coordinates of any straight line by $\lambda\mu\nu\rho$.

If α, β be the Cartesian coordinates of the centre of any circle whose power-coordinate with respect to a circle be ξ; if a, b be the Cartesian coordinates of the centre of the latter; and R, r be the radii of the two circles : we shall have

$$\xi \propto (\alpha-a)^2+(\beta-b)^2-\mathrm{R}^2-r^2 ;$$

so that the power-coordinates of any *circle* are quadric functions of a particular form of the Cartesian coordinates of the centre of the circle.

Similarly, if x be the power-coordinate of a point whose Cartesian coordinates are α, β :

$$x \propto (\alpha-a)^2+(\beta-b)^2-r^2 ;$$

or the power-coordinates of a *point* are quadric functions of a particular form of the Cartesian coordinates of the point.

In the case of a straight line, whose Cartesian equation is

$$x \cos \alpha+y \sin \alpha=p,$$

we shall have

$$\lambda \propto p-a \cos \alpha-b \sin \alpha.$$

Thus the power-coordinates of a *straight line* are linear functions of a particular form of what may be called the Cartesian coordinates of the straight line.

48. If θ denote as previously the line at infinity, and the system of reference be denoted by $(1, 2, 3, 4)$, then P being any point; since $\pi_{P,\theta}=1$, $\pi_{P,P}=0$, we see that the coordinates of P must satisfy :—

i. a homogeneous quadric relation,

$$\Pi\begin{pmatrix} P, 1, 2, 3, 4 \\ P, 1, 2, 3, 4 \end{pmatrix}=0 \; ;$$

ii. a non-homogeneous linear relation,

$$\Pi\begin{pmatrix} P, 1, 2, 3, 4 \\ \theta, 1, 2, 3, 4 \end{pmatrix}=0.$$

Let us suppose that $xyzw$, the coordinates of P, are given by

$$x=k_1.\pi_{P,1} \; ; \quad y=k_2.\pi_{P,2} \; ; \quad z=k_3.\pi_{P,3} \; ; \quad w=k_4.\pi_{P,4} \; ;$$

then $xyzw$ must satisfy the relation, which is called the *absolute* :—

$$\psi(x, y, z, w) \equiv \begin{vmatrix} 0, & \dfrac{x}{k_1}, & \dfrac{y}{k_2}, & \dfrac{z}{k_3}, & \dfrac{w}{k_4} \\ \dfrac{x}{k_1}, & \pi_{1,1}, & \pi_{1,2}, & \pi_{1,3}, & \pi_{1,4} \\ \dfrac{y}{k_2}, & \pi_{2,1}, & \pi_{2,2}, & \pi_{2,3}, & \pi_{2,4} \\ \dfrac{z}{k_3}, & \pi_{3,1}, & \pi_{3,2}, & \pi_{3,3}, & \pi_{3,4} \\ \dfrac{w}{k_4}, & \pi_{4,1}, & \pi_{4,2}, & \pi_{4,3}, & \pi_{4,4} \end{vmatrix} =0. \quad \ldots \quad (69)$$

Then the relation (ii) may be written

or

$$\left. \begin{aligned} k_1\frac{\partial\psi}{\partial x}+k_2\frac{\partial\psi}{\partial y}+k_3\frac{\partial\psi}{\partial z}+k_4\frac{\partial\psi}{\partial w}=K, \\ x\frac{\partial\psi}{\partial k_1}+y\frac{\partial\psi}{\partial k_2}+z\frac{\partial\psi}{\partial k_3}+w\frac{\partial\psi}{\partial k_4}=K \; ; \end{aligned} \right\} \quad \ldots \quad \ldots \quad (70)$$

where K is some constant.[*]

49. Similarly if S denote any circle, since $\pi_{S,\theta}=1$, we see that its coordinates $(\xi, \eta, \zeta, \omega)$ must satisfy the linear relation

$$\Pi\begin{pmatrix} S, 1, 2, 3, 4 \\ \theta, 1, 2, 3, 4 \end{pmatrix}=0,$$

or

$$\xi\frac{\partial\psi}{\partial k_1}+\eta\frac{\partial\psi}{\partial k_2}+\zeta\frac{\partial\psi}{\partial k_3}+\eta\frac{\partial\psi}{\partial k_4}=K . \quad \ldots \quad \ldots \quad (71)$$

* [If we write $\psi (x, y, z, w) \equiv (a_{1,1}, a_{1,2}, \ldots) (x, y, z, w)^2$, then we shall have,

$$\frac{K}{\Pi\begin{pmatrix} 1, 2, 3, 4 \\ 1, 2, 3, 4 \end{pmatrix}}=\frac{2a_{1,1} \cdot k_1^2}{\Pi\begin{pmatrix} 2, 3, 4 \\ 2, 3, 4 \end{pmatrix}}=\&c.—October, 1886.]$$

50. Again if L denote any straight line, since $\pi_{L,\theta}=0$, and $\pi_{L,L}=-2$; we see that its coordinates $(\lambda, \mu, \nu, \rho)$ must satisfy, (i) the homogeneous linear relation,

$$\Pi\begin{pmatrix} L, 1, 2, 3, 4 \\ \theta, 1, 2, 3, 4 \end{pmatrix}=0,$$

or

$$\lambda\frac{\partial\psi}{\partial k_1}+\mu\frac{\partial\psi}{\partial k_2}+\nu\frac{\partial\psi}{\partial k_3}+\rho\frac{\partial\psi}{\partial k_4}=0; \quad . \quad . \quad . \quad . \quad . \quad . \quad (72)$$

and (ii) the non-homogeneous quadric relation,

$$\Pi\begin{pmatrix} L, 1, 2, 3, 4 \\ L, 1, 2, 3, 4 \end{pmatrix}=0,$$

or

$$\psi(\lambda, \mu, \nu, \rho)=-K. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (73)$$

The Circle.—§§ 51–55.

51. Let P be any point on a circle S, then $\pi_{S,P}=0$; hence the equation

$$\Pi\begin{pmatrix} P, 1, 2, 3, 4 \\ S, 1, 2, 3, 4 \end{pmatrix}=0,$$

leads to

$$\frac{\partial\psi}{\partial\xi}x+\frac{\partial\psi}{\partial\eta}y+\frac{\partial\psi}{\partial\zeta}z+\frac{\partial\psi}{\partial\omega}w=0. \quad . \quad . \quad . \quad . \quad . \quad (74)$$

Thus the equation of a circle is of the first degree.

It follows that the general equation of the first degree

$$ax+by+cz+dw=0$$

represents in general a circle, whose coordinates are given by,

$$\frac{\frac{\partial\psi}{\partial\xi}}{a}=\frac{\frac{\partial\psi}{\partial\eta}}{b}=\frac{\frac{\partial\psi}{\partial\zeta}}{c}=\frac{\frac{\partial\psi}{\partial\omega}}{d}=\frac{K}{ak_1+bk_2+ck_3+dk_4}, \quad . \quad . \quad . \quad . \quad (75)$$

by equation (71).

52. Given any two circles whose coordinates are $(\xi, \eta, \zeta, \omega)$, $(\xi', \eta', \zeta', \omega')$; their power π is given by

$$\Pi\begin{pmatrix} S, 1, 2, 3, 4 \\ S', 1, 2, 3, 4 \end{pmatrix}=0,$$

or

$$\left.\begin{aligned} \pi.K &=\xi\frac{\partial\psi}{\partial\xi}+\eta\frac{\partial\psi}{\partial\eta}+\zeta'\frac{\partial\psi}{\partial\zeta}+\omega'\frac{\partial\psi}{\partial\omega} \\ &=\xi\frac{\partial\psi}{\partial\xi'}+\eta\frac{\partial\psi}{\partial\eta'}+\zeta\frac{\partial\psi}{\partial\zeta'}+\omega\frac{\partial\psi}{\partial\omega'} \end{aligned}\right\} . \quad . \quad . \quad . \quad . \quad (76)$$

In particular the radius of the circle $\xi\eta\zeta\omega$ is given by

$$-2r^2 . \mathrm{K} = \xi\frac{\partial\psi}{\partial\xi} + \eta\frac{\partial\psi}{\partial\eta} + \zeta\frac{\partial\psi}{\partial\zeta} + \omega\frac{\partial\psi}{\partial\omega},$$

or

$$-r^2 . \mathrm{K} = \psi(\xi,\ \eta,\ \zeta,\ \omega). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (77)$$

53. Hence the radius of the circle

$$ax + by + cz + dw = 0,$$

is given by

$$\begin{vmatrix} a_{1,1}, & a_{1,2}, & a_{1,3}, & a_{1,4}, & a \\ a_{2,1}, & a_{2,2}, & a_{2,3}, & a_{2,4}, & b \\ a_{3,1}, & a_{3,2}, & a_{3,3}, & a_{3,4}, & c \\ a_{4,1}, & a_{4,2}, & a_{4,3}, & a_{4,4}, & d \\ a, & b, & c, & d, & \mathrm{M}r^2 \end{vmatrix} = 0 ; \quad . \quad . \quad . \quad . \quad (78)$$

where

$$\mathrm{M} = -\frac{4}{\mathrm{K}}(ak_1 + bk_2 + ck_3 + dk_4)^2,$$

and $a_{1,1}$, $a_{1,2}$, &c., are the coefficients in the equation to the absolute, so that

$$\psi(xyzw) \equiv a_{1,1}\,x^2 + a_{2,2}\,y^2 + \ldots + 2a_{1,2}\,xy + \ldots = 0.$$

54. Again the power of the circle $(\xi\eta\zeta\omega)$ with respect to the circle

$$ax + by + cz + dw = 0,$$

is clearly given by

$$\pi = \frac{a\xi + b\eta + c\zeta + d\omega}{ak_1 + bk_2 + ck_3 + dk_4} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (79)$$

55. And again the power of the two circles

$$ax + by + cz + dw = 0,$$
$$a'x + b'y + c'z + d'w = 0,$$

is given by

$$\begin{vmatrix} a_{1,1}, & a_{1,2}, & a_{1,3}, & a_{1,4}, & a \\ a_{2,1}, & a_{2,2}, & a_{2,3}, & a_{2,4}, & b \\ a_{3,1}, & a_{3,2}, & a_{3,3}, & a_{3,4}, & c \\ a_{4,1}, & a_{4,2}, & a_{4,3}, & a_{4,4}, & d \\ a', & b', & c', & d', & \mathrm{M}\pi \end{vmatrix} = 0 ;$$

where

$$\mathrm{M} = \frac{2}{\mathrm{K}}(ak_1 + bk_2 + ck_3 + dk_4)(a'k_1 + b'k_2 + c'k_3 + d'k_4). \quad . \quad . \quad . \quad (80)$$

Whence, if the two circles cut at an angle ϕ, we have

$$\cos \phi = -\frac{a'\dfrac{\partial \Psi}{\partial a} + b'\dfrac{\partial \Psi}{\partial b} + c'\dfrac{\partial \Psi}{\partial c} + d'\dfrac{\partial \Psi}{\partial d}}{2\sqrt{\Psi(a, b, c, d) . \Psi(a', b', c', d')}} \; ; \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (81)$$

where

$$\Psi(a, b, c, d) \equiv - \begin{vmatrix} a_{1,1}, & a_{1,2}, & a_{1,3}, & a_{1,4}, & a \\ a_{2,1}, & a_{2,2}, & a_{2,3}, & a_{2,4}, & b \\ a_{3,1}, & a_{3,2}, & a_{3,3}, & a_{3,4}, & c \\ a_{4,1}, & a_{4,2}, & a_{4,3}, & a_{4,4}, & d \\ a, & b, & c, & d, & 0 \end{vmatrix} .$$

The straight Line.—§§ 56–58.

56. Proceeding as in § 51, we see that the equation to the straight line, whose coordinates are $(\lambda, \mu, \nu, \rho)$, is

$$\frac{\partial \psi}{\partial \lambda} x + \frac{\partial \psi}{\partial \mu} y + \frac{\partial \psi}{\partial \nu} z + \frac{\partial \psi}{\partial \rho} w = 0. . \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (82)$$

But by equation (72)

$$\frac{\partial \psi}{\partial \lambda} k_1 + \frac{\partial \psi}{\partial \mu} k_2 + \frac{\partial \psi}{\partial \nu} k_3 + \frac{\partial \psi}{\partial \rho} k_4 = 0 \; ;$$

hence the equation

$$ax + by + cz + dw = 0,$$

will represent a straight line, provided that

$$ak_1 + bk_2 + ck_3 + dk_4 = 0 \; ; \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (83)$$

and if this condition be satisfied the coordinates of the lines are given by,

$$\frac{\dfrac{\partial \psi}{\partial \lambda}}{a} = \frac{\dfrac{\partial \psi}{\partial \mu}}{b} = \frac{\dfrac{\partial \psi}{\partial \nu}}{c} = \frac{\dfrac{\partial \psi}{\partial \rho}}{d} = 2\sqrt{\frac{-\mathrm{K} . \Delta}{\Psi}} \; ; \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (84)$$

where

$$\Delta \equiv \begin{vmatrix} a_{1,1}, & a_{1,2}, & a_{1,3}, & a_{1,4} \\ a_{2,1}, & a_{2,2}, & a_{2,3}, & a_{2,4} \\ a_{3,1}, & a_{3,2}, & a_{3,3}, & a_{3,4} \\ a_{4,1}, & a_{4,2}, & a_{4,3}, & a_{4,4} \end{vmatrix} ,$$

and Ψ denotes the same expression as in § 55.

57. The power of the straight line

$$ax+by+cz+dw=0,$$

and the circle $(\xi,\ \eta,\ \zeta,\ \omega)$ is given by,

$$\pi\mathrm{K}=\xi\frac{\partial\psi}{\partial\lambda}+\eta\frac{\partial\psi}{\partial\mu}+\zeta\frac{\partial\psi}{\partial\nu}+\omega\frac{\partial\psi}{\partial\rho}$$

$$=2\sqrt{\frac{-\mathrm{K}.\Delta}{\Psi}}\cdot(a\xi+b\eta+c\zeta+d\omega)\ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ (85)$$

And the loci represented by the equations

$$ax+by+cz+dw=0,$$
$$a'x+b'y+c'z+d'w=0,$$

intersect at the angle ϕ, given by

$$\cos\phi=-\tfrac{1}{2}\frac{a'\dfrac{\partial\Psi}{\partial a}+b'\dfrac{\partial\Psi}{\partial b}+c'\dfrac{\partial\Psi}{\partial c}+d'\dfrac{\partial\Psi}{\partial d}}{\sqrt{\Psi(a,\,b,\,c,\,d).\Psi(a',\,b',\,c',\,d')}}\cdot\ \ .\ \ .\ \ .\ \ .\ \ .\ \ (86)$$

58. The coordinates of the line at infinity are k_1, k_2, k_3, k_4; hence the equation to the line at infinity is

$$x\frac{\partial\psi}{\partial k_1}+y\frac{\partial\psi}{\partial k_2}+z\frac{\partial\psi}{\partial k_3}+w\frac{\partial\psi}{\partial k_4}=0.$$

The Point.—§§ 59–61.

59. The power of the point $(xyzw)$ with respect to the circle

$$ax+by+cz+dw=0,$$

is

$$\frac{ax+by+cz+dw}{ak_1+bk_2+ck_3+dk_4}\cdot$$

But if the equation represents a straight line, then we see that the perpendicular on it from the point

$$=(ax+by+cz+dw)\sqrt{\frac{-\Delta}{\mathrm{K}\Psi}}\cdot\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ (87)$$

60. The power of the two points $(xyzw)$, $(x'y'z'w')$ is given by equation (76)

$$\pi\mathrm{K}=x\frac{\partial\psi}{\partial x'}+y\frac{\partial\psi}{\partial y'}+z\frac{\partial\psi}{\partial z'}+w\frac{\partial\psi}{\partial w'}.$$

Hence, if

$$\psi(xyzw) = a_{1,1}x^2 + 2a_{1,2}xy + \ldots ;$$

the distance δ between the points is given by,

$$-\delta^2.\mathrm{K} = a_{1,1}(x-x')^2 + 2a_{1,2}(x-x')(y-y') + \ldots$$
$$= \psi\{x-x',\ y-y',\ z-z',\ w-w'\}. \qquad \ldots \qquad (88)$$

Thus, if δs be a small element of arc, we shall have

$$-\mathrm{K}.(\delta s)^2 = \psi\{\delta x,\ \delta y,\ \delta z,\ \delta w\}. \qquad \ldots \qquad (89)$$

61. If P, Q, R be any three points, then by equation (15) we have for the area of the triangle

$$-16\{\Delta(\mathrm{P,\ Q,\ R})\}^2 = \Pi\begin{pmatrix} \theta,\ \mathrm{P,\ Q,\ R} \\ \theta,\ \mathrm{P,\ Q,\ R} \end{pmatrix};$$

and by § 8

$$\Pi\begin{pmatrix} \theta,\ \mathrm{P,\ Q,\ R} \\ \theta,\ \mathrm{P,\ Q,\ R} \end{pmatrix} \times \Pi\begin{pmatrix} 1,\ 2,\ 3,\ 4 \\ 1,\ 2,\ 3,\ 4 \end{pmatrix} = \left\{\Pi\begin{pmatrix} \theta,\ \mathrm{P,\ Q,\ R} \\ 1,\ 2,\ 3,\ 4 \end{pmatrix}\right\}^2.$$

Hence if $(x_1 y_1 z_1 w_1)$, $(x_2 y_2 z_2 w_2)$, $(x_3 y_3 z_3 w_3)$ be the coordinates of the points P, Q, R referred to any system of circles (1, 2, 3, 4) we shall have

$$\Delta(\mathrm{P,\ Q,\ R}) = \mu \begin{vmatrix} x_1, & x_2, & x_3, & k_1 \\ y_1, & y_2, & y_3, & k_2 \\ z_1, & z_2, & z_3, & k_3 \\ w_1, & w_2, & w_3, & k_4 \end{vmatrix}; \qquad \ldots \qquad (90)$$

where,

$$-4\mu^2 k_1^2 k_2^2 k_3^2 k_4^2.\Pi\begin{pmatrix} 1,\ 2,\ 3,\ 4 \\ 1,\ 2,\ 3,\ 4 \end{pmatrix} = 1.$$

Coordinate Systems of Reference.—§§ 62–66.

62. There are two systems of circles which are convenient as systems of reference— (i.) a system consisting of four circles cutting one another orthogonally,* (ii.) a system of two circles cutting orthogonally, and their two points of intersection. The former has been called the "orthogonal" system, and was first used by DARBOUX, 'Sur une Classe remarquable de Courbes et de Surfaces algébriques' (Note X., 1873). The latter system might be called the "semi-orthogonal" system; it is mentioned by Mr. HOMERSHAM Cox in the paper "On Systems of Circles and Bicircular Quartics" ('Quart. Journ. Math.,' vol. 19, 1883, p. 116).

* [CASEY uses five orthogonal spheres—"Cyclides" (1871), p. 600. But the first use of four mutually orthotomic circles was, I believe, by CLIFFORD in a series of questions proposed by him in the 'Educational Times' for 1865–6. See Reprint, vol. 6.—October, 1886.]

63. In the case of the orthogonal system it is most convenient to take the constants k_1, k_2, k_3, k_4 equal to the reciprocals of the four radii, so that the equation to the absolute is

and
$$\left.\begin{array}{c} \psi(x,\ y,\ z,\ w) \equiv x^2 + y^2 + z^2 + w^2 = 0 \\ \mathrm{K} = -4 \\ \Psi(a,\ b,\ c,\ d) = a^2 + b^2 + c^2 + d^2 \end{array}\right\} \quad \ldots \ldots \quad (91)$$

64. In the case of the semi-orthogonal system (see § 29), if r_1, r_2 be the radii of the circles, e the distance between their points of intersection, it is convenient to take $k_1 = \dfrac{1}{r_1}$, $k_2 = \dfrac{1}{r_2}$, $k_3 = k_4 = \dfrac{1}{e}$.

We shall have

$$\left.\begin{array}{c} \psi(x,\ y,\ z,\ w) \equiv x^2 + y^2 - 4zw \\ \mathrm{K} = -4 \\ \Psi(a,\ b,\ c,\ d) = a^2 + b^2 - cd \end{array}\right\} \quad \ldots \ldots \quad (92)$$

65. Thus the angle ϕ at which the loci

$$ax + by + cz + dw = 0,$$
$$a'x + b'y + c'z + d'w = 0,$$

intersect, is given by

$$\cos\phi = -\frac{aa' + bb' + cc' + dd'}{\sqrt{(a^2 + b^2 + c^2 + d^2)(a'^2 + b'^2 + c'^2 + d'^2)}},$$

or

$$\cos\phi = -\frac{aa' + bb' - \frac{1}{2}(cd' + c'd)}{\sqrt{(a^2 + b^2 - cd)(a'^2 + b'^2 - c'd')}};$$

according as the system of reference is the "orthogonal" or the "semi-orthogonal" system.

66. Occasionally it may not be convenient to take for system of reference either of the systems just considered. In some cases, however, the equations may be simplified by referring the coordinates of a point to one system of circles, and the coordinates of any line or circle to the system cutting the former system orthogonally. Thus, if the system of reference be (1, 2, 3, 4), and (5, 6, 7, 8) denote the system orthogonal to this, then taking k_1, k_2, k_3, k_4 equal to unity, the equation of the circle (or line), whose coordinates referred to (5, 6, 7, 8) are ξ, η, ζ, ω, referred to the system (1, 2, 3, 4) is

$$\frac{x\xi}{\pi_{1,5}} + \frac{\eta y}{\pi_{2,6}} + \frac{z\zeta}{\pi_{3,7}} + \frac{w\omega}{\pi_{4,8}} = 0.$$

The equation

$$ax + by + cz + dw = 0$$

will represent a circle whose coordinates are

$$\xi = \frac{a\pi_{1,5}}{a+b+c+d}, \quad \eta = \&\text{c. ;}$$

unless $a+b+c+d=0$, in which case the equation represents a straight line.

Inversion.—§§ 67, 68.

67. Let $xyzw$ be the power-coordinates of any point P with respect to the system of circles $(1, 2, 3, 4)$; let P$'$ be the inverse of P with respect to any point O ; then since by § 3, $\dfrac{\pi_{1,\text{P}}}{\sqrt{\pi_{0,1}.\pi_{0,\text{P}}}}$ is unaltered by inversion, it follows that, if XYZW be the co-ordinates of P$'$ referred to the system which is the inverse of $(1, 2, 3, 4)$, we must have

$$x=\alpha\text{X}, \quad y=\beta\text{Y}, \quad z=\gamma\text{Z}, \quad w=\delta\text{W},$$

where $\alpha, \beta, \gamma, \delta$ are some constants.

Thus if the equation in power-coordinates of any curve be $f(xyzw)=0$, the equation to the inverse curve will be $f(\alpha\text{X}, \beta\text{Y}, \gamma\text{Z}, \delta\text{W})=0$.

68. The system consisting of two rectangular axes, the point of intersection, and the line at infinity, is clearly the inverse of a system of two orthogonal circles, and their two points of intersection, the centre of inversion being one of these points.

For instance, the equation of a parabola expressed in power-coordinates is clearly

$$\text{X}^2=2a\text{YZ.}$$

Hence the equation to the inverse of a parabola is of the form

$$x^2=2ayz,$$

x, y having reference to the orthogonal circles, z, w to their two points of inter-section.

Similarly the equation to the inverse of a central conic must be of the form

$$\alpha x^2+\beta y^2=\gamma z^2,$$

α, β having the same or different signs according as the conic is an ellipse or hyperbola.

Chapter VI.—General Equation of the Second Degree in Power-Coordinates.

Nature of the Curve.—§§ 69, 70.

69. The most general equation of the second degree in power-coordinates may be written

$$\phi(x, y, z, w) \equiv ax^2 + by^2 + cz^2 + dw^2 + 2fyz + 2gzx + 2hxy + 2lxw + 2myw + 2nzw = 0, \quad (93)$$

$(xyzw)$ being the coordinates of a point on the curve, referred to some system of circles, and therefore satisfying the equation of the absolute ψ, which is also of the second degree. Consequently the form (93) contains only eight arbitrary constants.

Now $xyzw$ may be expressed as linear functions of $X^2 + Y^2$, $X, Y, 1$; (X, Y) being the Cartesian coordinates of the point; and substituting, it is easily seen that (93) may be expressed in the form

$$(X^2 + Y^2)^2 + U_1(X^2 + Y^2) + U_2 = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (94)$$

U_1, U_2 being of the first and second degree; this equation contains eight constants, and since (94) represents a curve having nodes at each of the circular points at infinity, it appears that (93) is a form to which every bicircular quartic can be reduced.

70. It is otherwise evident that, since the equation of a straight line is of the first degree, every straight line cuts ϕ in four points, unless ϕ is satisfied by the coordinates of the line at infinity; for these coordinates satisfy the equation of every straight line, and therefore in this case ϕ must represent a circular cubic and the line at infinity.

Equation to Tangent at any Point.—§§ 71, 72.

71. Let $(\xi \eta \zeta \omega)$ be the coordinates of any circle touching the curve ϕ at the point $(x'y'z'w')$. We must have, by equation (74)

$$\frac{\partial \psi}{\partial \xi} x' + \frac{\partial \psi}{\partial \eta} y' + \frac{\partial \psi}{\partial \zeta} z' + \frac{\partial \psi}{\partial \omega} w' = 0,$$

and since this passes through the point $(x' + \delta x', y' + \delta y', z' + \delta z', w' + \delta w')$ we must have

$$\frac{\partial \psi}{\partial \xi} \delta x' + \frac{\partial \psi}{\partial \eta} \delta y' + \frac{\partial \psi}{\partial \zeta} \delta z' + \frac{\partial \psi}{\partial \omega} \delta w' = 0;$$

also

$$\frac{\partial \psi}{\partial x'}\, \delta x' + \frac{\partial \psi}{\partial y'}\, \delta y' + \frac{\partial \psi}{\partial z'}\, \delta z' + \frac{\partial \psi}{\partial w'}\, \delta w' = 0,$$

$$\frac{\partial \phi}{\partial x'}\, \delta x' + \frac{\partial \phi}{\partial y'}\, \delta y' + \frac{\partial \phi}{\partial z'}\, \delta z' + \frac{\partial \phi}{\partial w'}\, \delta w' = 0.$$

Hence we must have

$$\frac{\dfrac{\partial \psi}{\partial \xi}}{\dfrac{\partial \phi}{\partial x'} + k \dfrac{\partial \psi}{\partial x'}} = \frac{\dfrac{\partial \psi}{\partial \eta}}{\dfrac{\partial \phi}{\partial y'} + k \dfrac{\partial \psi}{\partial y'}} = \frac{\dfrac{\partial \psi}{\partial \zeta}}{\dfrac{\partial \phi}{\partial z'} + k \dfrac{\partial \psi}{\partial z'}} = \frac{\dfrac{\partial \psi}{\partial \omega}}{\dfrac{\partial \phi}{\partial w'} + k \dfrac{\partial \psi}{\partial w'}}\, ; \quad \cdots \quad (95)$$

and every circle whose coordinates satisfy these equations must touch the curve $\phi = 0$, at the point $(x'y'z'w')$.

72. Let $(\lambda \mu \nu \rho)$ be the coordinates of the tangent at the point $(x'y'z'w')$, then we must have

$$\frac{\dfrac{\partial \psi}{\partial \lambda}}{\dfrac{\partial \phi}{\partial x'} + k \dfrac{\partial \psi}{\partial x'}} = \frac{\dfrac{\partial \psi}{\partial \mu}}{\dfrac{\partial \phi}{\partial y'} + k \dfrac{\partial \psi}{\partial y'}} = \frac{\dfrac{\partial \psi}{\partial \nu}}{\dfrac{\partial \phi}{\partial z'} + k \dfrac{\partial \psi}{\partial z'}} = \frac{\dfrac{\partial \psi}{\partial \rho}}{\dfrac{\partial \phi}{\partial w'} + k \dfrac{\partial \psi}{\partial w'}}\, ; \quad \cdots \quad (96)$$

but, by § 56, we must have

$$k_1 \frac{\partial \psi}{\partial \lambda} + k_2 \frac{\partial \psi}{\partial \mu} + k_3 \frac{\partial \psi}{\partial \nu} + k_4 \frac{\partial \psi}{\partial \rho} = 0,$$

where $(k_1,\ k_2,\ k_3,\ k_4)$ are the coordinates of the line at infinity; hence, if k be determined by the equation

$$\left(k_1 \frac{\partial}{\partial x'} + k_2 \frac{\partial}{\partial y'} + k_3 \frac{\partial}{\partial z'} + k_4 \frac{\partial}{\partial w'} \right)(\phi + k\psi) = 0\, ; \quad \cdots \quad (97)$$

then the coordinates of the tangent to the curve are given by (96), and the equation to it is

$$\left(x \frac{\partial}{\partial x'} + y \frac{\partial}{\partial y'} + z \frac{\partial}{\partial z'} + w \frac{\partial}{\partial w'} \right)(\phi + k\psi) = 0,$$

or

$$\left(k_1 \frac{\partial \phi}{\partial x'} + k_2 \frac{\partial \phi}{\partial y'} + k_3 \frac{\partial \phi}{\partial z'} + k_4 \frac{\partial \phi}{\partial w'} \right)\left(x \frac{\partial \psi}{\partial x'} + y \frac{\partial \psi}{\partial y'} + z \frac{\partial \psi}{\partial z'} + w \frac{\partial \psi}{\partial w'} \right)$$

$$= \left(k_1 \frac{\partial \psi}{\partial x'} + k_2 \frac{\partial \psi}{\partial y'} + k_3 \frac{\partial \psi}{\partial z'} + k_4 \frac{\partial \psi}{\partial w'} \right)\left(x \frac{\partial \phi}{\partial x'} + y \frac{\partial \phi}{\partial y'} + z \frac{\partial \phi}{\partial z'} + w \frac{\partial \phi}{\partial w'} \right). \quad \cdot \quad (98)$$

Bitangent Circles.—§§ 73–75.

73. A circle

$$\left(x\frac{\partial}{\partial x'}+y\frac{\partial}{\partial y'}+z\frac{\partial}{\partial z'}+w\frac{\partial}{\partial w'}\right)(\phi+k\psi)=0,$$

will clearly touch the curve ϕ at the point x'', y'', z'', w'' if

$$\frac{\dfrac{\partial\phi}{\partial x'}+k\dfrac{\partial\psi}{\partial x'}}{\dfrac{\partial\phi}{\partial x''}+k\dfrac{\partial\psi}{\partial x''}}=\frac{\dfrac{\partial\phi}{\partial y'}+k\dfrac{\partial\psi}{\partial y'}}{\dfrac{\partial\phi}{\partial y''}+k\dfrac{\partial\psi}{\partial y''}}=\frac{\dfrac{\partial\phi}{\partial z'}+k\dfrac{\partial\psi}{\partial z'}}{\dfrac{\partial\phi}{\partial z''}+k\dfrac{\partial\psi}{\partial z''}}=\frac{\dfrac{\partial\phi}{\partial w'}+k\dfrac{\partial\psi}{\partial w'}}{\dfrac{\partial\phi}{\partial w''}+k\dfrac{\partial\psi}{\partial w''}};$$

i.e., if k satisfy the equation

$$H(\phi+k\psi)=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (99)$$

where $H(u)$ denotes the Hessian of u.

Since this equation is of the fourth degree in k, we infer that there are in general four systems of bitangent circles, each circle belonging to any system cutting a certain fixed circle orthogonally, the coordinates of this circle being proportional to the minors of the constituents of any row of the determinant $H(\phi+k\psi)$.

74. If the coordinates of a bitangent circle satisfy the condition which must be satisfied by coordinates of any straight line, the corresponding equation will represent the double tangents from the centre of the corresponding circle. In general, then, there are eight double tangents.

75. It is clear that, if by any linear transformation of coordinates the equations $\phi=0$, $\psi=0$ become respectively $\Phi=0$, $\Psi=0$, then the same value of k must satisfy both

$$H(\phi+k\psi)=0 \quad \text{and} \quad H(\Phi+k\Psi)=0.$$

Hence the coefficients of the powers of k in equation (99) are invariants.

Equation to Normal at any Point.—§§ 76–79.

76. Let $(\xi\eta\zeta\omega)$ be the coordinates of any circle which cuts the curve $\phi(xyzw)=0$, orthogonally at the point $(x'y'z'w')$, then by equation (75) we must have

$$\left(\xi\frac{\partial}{\partial x'}+\eta\frac{\partial}{\partial y'}+\zeta\frac{\partial}{\partial z'}+\omega\frac{\partial}{\partial w'}\right)(\phi+k\psi)=0, \quad . \quad . \quad . \quad . \quad . \quad (100)$$

for all values of k.

77. It follows that if $(\lambda, \mu, \nu, \rho)$ be the coordinates of the normal at $(x'y'z'w')$ we must have

$$\lambda\frac{\partial\phi}{\partial x'}+\mu\frac{\partial\phi}{\partial y'}+\nu\frac{\partial\phi}{\partial z'}+\rho\frac{\partial\phi}{\partial w'}=0,$$

$$\lambda\frac{\partial\psi}{\partial x'}+\mu\frac{\partial\psi}{\partial y'}+\nu\frac{\partial\psi}{\partial z'}+\rho\frac{\partial\psi}{\partial w'}=0,$$

and

$$\lambda\frac{\partial\psi}{\partial k_1}+\mu\frac{\partial\psi}{\partial k_2}+\nu\frac{\partial\psi}{\partial k_3}+\rho\frac{\partial\psi}{\partial k_4}=0\ ;$$

(k_1, k_2, k_3, k_4) being the coordinates of the line at infinity. Hence the equation to the normal is

$$\begin{vmatrix} \dfrac{\partial\psi}{\partial x}, & \dfrac{\partial\psi}{\partial y}, & \dfrac{\partial\psi}{\partial z}, & \dfrac{\partial\psi}{\partial w} \\[2ex] \dfrac{\partial\phi}{\partial x''}, & \dfrac{\partial\phi}{\partial y''}, & \dfrac{\partial\phi}{\partial z''}, & \dfrac{\partial\phi}{\partial w'} \\[2ex] \dfrac{\partial\psi}{\partial x''}, & \dfrac{\partial\psi}{\partial y''}, & \dfrac{\partial\psi}{\partial z''}, & \dfrac{\partial\psi}{\partial w'} \\[2ex] \dfrac{\partial\psi}{\partial k_1}, & \dfrac{\partial\psi}{\partial k_2}, & \dfrac{\partial\psi}{\partial k_3}, & \dfrac{\partial\psi}{\partial k_4} \end{vmatrix}=0. \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (101)$$

78. We can easily deduce from equation (101) that normals can be drawn from the point $(x'y'z'w')$ to the curve $\phi=0$, at its points of intersection with the curve

$$\begin{vmatrix} \dfrac{\partial\phi}{\partial x}, & \dfrac{\partial\phi}{\partial y}, & \dfrac{\partial\phi}{\partial z}, & \dfrac{\partial\phi}{\partial w} \\[2ex] \dfrac{\partial\psi}{\partial x}, & \dfrac{\partial\psi}{\partial y}, & \dfrac{\partial\psi}{\partial z}, & \dfrac{\partial\psi}{\partial w} \\[2ex] \dfrac{\partial\psi}{\partial x''}, & \dfrac{\partial\psi}{\partial y''}, & \dfrac{\partial\psi}{\partial z''}, & \dfrac{\partial\psi}{\partial w'} \\[2ex] \dfrac{\partial\psi}{\partial k_1}, & \dfrac{\partial\psi}{\partial k_2}, & \dfrac{\partial\psi}{\partial k_3}, & \dfrac{\partial\psi}{\partial k_4} \end{vmatrix}=0. \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (102)$$

This curve is clearly of the second degree, but since it is satisfied by (k_1, k_2, k_3, k_4) the coordinates of the line at infinity, it represents a circular cubic. Hence, in general, eight normals can be drawn from any given point to the curve.

79. In the case of a circle cutting the curve $\phi=0$, normally at the point $(x'y'z'w')$, we shall have

$$x'\frac{\partial\phi}{\partial\xi}+y'\frac{\partial\phi}{\partial\eta}+z'\frac{\partial\phi}{\partial\zeta}+w'\frac{\partial\phi}{\partial\omega}=0,$$

$$x'\frac{\partial\psi}{\partial\xi}+y'\frac{\partial\psi}{\partial\eta}+z'\frac{\partial\psi}{\partial}+w'\frac{\partial\psi}{\partial\omega}=0.$$

If then $(\xi\eta\zeta\omega)$ be chosen so that

$$\frac{\dfrac{\partial\phi}{\partial\xi}}{\dfrac{\partial\psi}{\partial\xi}}=\frac{\dfrac{\partial\phi}{\partial\eta}}{\dfrac{\partial\psi}{\partial\eta}}=\frac{\dfrac{\partial\phi}{\partial\zeta}}{\dfrac{\partial\psi}{\partial\zeta}}=\frac{\dfrac{\partial\phi}{\partial\omega}}{\dfrac{\partial\psi}{\partial\omega}}=-k \text{ say,}$$

the circle $(\xi\eta\zeta\omega)$ will cut the curve ϕ orthogonally in four points, and we see at once, that k must satisfy the equation

$$\mathrm{H}(\phi+k\psi)=0 ;$$

and then $(\xi\eta\zeta\omega)$ are proportional to the minors of the constituents of any row in the determinant $\mathrm{H}(\phi+k\psi)$.

Thus it appears there are four circles which cut the curve $\phi=0$ orthogonally : and these circles are identical with the four which are mentioned in § 73, as being orthogonal respectively to the four systems of bitangent circles.

The Principal Circles.—§§ 80–82.

80. The four circles considered in § 79 have been called by MOUTARD the principal circles of the curve. And the curve may be considered as the envelope of a system of circles, which cut one of these principal circles orthogonally : it follows then that the curve is its own inverse with respect to any one of the principal circles (hence the principal circles must cut orthogonally) ; also the four points in which any principal circle cuts the curve must be cyclic points ; so that there are in general sixteen cyclic points.

81. We may prove independently that the principal circles cut orthogonally, thus ; taking for our system of reference an orthogonal system, so that the equation of the absolute is

$$\psi\equiv x^2+y^2+z^2+w^2=0 ;$$

then the coordinates of any principal circle being $(\xi\eta\zeta\omega)$ we must have, if

$$\phi\equiv ax^2+by^2+cz^2+dw^2+2fyz+2gzx+2hxy+2lxw+2myw+2nzw=0 ;$$

$$\left.\begin{array}{l} a\xi+h\,\eta+g\zeta+l\,\omega=-k\xi \\ h\xi+b\,\eta+f\zeta+m\omega=-k\eta \\ g\xi+f\,\eta+c\,\zeta+n\,\omega=-k\zeta \\ l\,\xi+m\eta+n\zeta+d\,\omega=-k\omega \end{array}\right\}, \quad \cdots \cdots \cdots \quad (103)$$

where k is a root of the equation

$$\begin{vmatrix} a+k, & h, & g, & l \\ h, & b+k, & f, & m \\ g, & f, & c+k, & n \\ l, & m, & n, & d+k \end{vmatrix} = 0.$$

Let now $(\xi'\eta'\zeta'\omega')$ be the coordinates of the principal circle corresponding to another root k' of this equation; then multiplying equations (102) respectively by $(\xi'\eta'\zeta'\omega')$ and adding, we obtain at once

$$(k-k')(\xi\xi'+\eta\eta'+\zeta\zeta'+\omega\omega')=0.$$

Hence, if k, k' be unequal, the two circles must cut orthogonally.

82. If the curve have four principal circles—*i.e.*, if the roots of the discriminating quartic $H(\phi+k\psi)=0$ are all different, the curve cannot have a third double point—for, inverting with respect to any principal circle, the inverse must also be a double point, unless the point lies on the principal circle; since then a quartic curve can have but three double points, in the case of a bicircular quartic, the third double point must lie on each principal circle. Hence two of the roots of the discriminating quartic must be equal, and there are only two principal circles.

Similarly, if this third double point be a cusp, it is easy to see, by inverting with respect to a principal circle, that any circle touching the tangent to the cusp at the cusp must touch the principal circle; and hence there is only one principal circle, and the discriminating quartic must have three equal roots.

Reduction of General Equation.—§§ 83, 84.

83. If one of the principal circles be a circle of reference (say $x=0$), then it is clear that the terms involving xy, xz, xw, must be absent from the equation. Supposing, then, that the equation $H(\phi+k\psi)=0$ has all its roots unequal, then there are four principal circles, and taking these for circles of reference the equation must reduce to the form

$$ax^2+by^2+cz^2+dw^2=0.$$

Suppose, now, that two of the roots of the discriminant are equal; then taking the circles corresponding to the unequal roots, and two circles cutting them orthogonally as circles of reference, the equation will be of the form

$$ax^2+by^2+cz^2+dw^2+2nzw=0\;;$$

and the system of reference being orthogonal we have for the absolute

$$x^2+y^2+z^2+w^2=0\;;$$

and, therefore, the discriminant is

$$(k+a)(k+b)\{(k+c)(k+d)-n^2\}=0,$$

which can only have equal roots when

$$(c+d)^2-4cd+n^2=0.$$

If n be real this can only be satisfied by $n=0$, $c=d$.

But of our four circles of reference one must be imaginary; if one of the circles (x, y) is imaginary n will be real, and our equation will reduce to

$$ax^2+by^2+cz^2+cw^2=0,$$

which represents a pair of circles, and need not be considered.

If both circles (x, y) are real, then it is simplest to take as system of reference, these circles and their two points of intersection. So that the absolute is of the form

$$x^2+y^2=4zw,$$

and the discriminant becomes

$$(k+a)(k+b)\{cd-(2k-n)^2\}=0\;;$$

and, then, if this has equal roots, either $c=0$ or $d=0$; and the equation takes the form

$$ax^2+by^2+cz^2+2nzw=0,$$

which, by means of the absolute, may be written

$$ax^2+by^2+cz^2=0.$$

Let us suppose now that three of the roots of the discriminant are equal; referring our coordinates to the circle corresponding to the unequal root, and any three circles cutting it and one another orthogonally, the equation of the curve will reduce to

$$ax^2+by^2+cz^2+dw^2+2fyz+2myw+2nzw=0\;;$$

and the discriminating cubic is

$$(k+a)\begin{vmatrix} k+b, & f, & m \\ f, & k+c, & n \\ m, & n, & k+d \end{vmatrix}=0,$$

which we can easily prove can only have three equal roots when

$$f=m=n=0,\;\; b=c=d\;;$$

provided that f, m, n are all real, in which case the circle $x=0$ is imaginary.

In this case the curve takes the form

$$ax^2+by^2+bz^2+bw^2=0,$$

which represents a point.

If, however, $x=0$ be real, we may take as system of reference this circle, a circle cutting it orthogonally, and their two points of intersection; then, since the absolute is of the form

$$x^2+y^2=4zw,$$

the discriminant becomes

$$(k+a) \begin{vmatrix} k+b, & f, & m \\ f, & c, & n-2k \\ m, & n-2k, & d \end{vmatrix} =0 \; ;$$

which can only have three equal roots when $2b=-n$, $m=d=0$; in which case the equation of the curve takes the form

$$ax^2+by^2-4bzw+2fyz+cz^2=0,$$

and by taking instead of $y=0$, the circle $y+\lambda z$, which clearly cuts x orthogonally, we can get rid of the term cz^2.

And since

$$x^2+y^2=4zw,$$

this equation can be further reduced to the form

$$ax^2+2fyz=0.$$

84. Thus we see that the equation of a bi-circular quartic can be reduced to one of three forms :—

$$\text{(A.)} \qquad ax^2+by^2+cz^2+dw^2=0,$$

in which case there are four principal circles, the equation of the absolute being

$$x^2+y^2+z^2+w^2=0.$$

$$\text{(B.)} \qquad ax^2+by^2+cz^2=0,$$

the equation of the absolute being

$$x^2+y^2=4zw,$$

in which case there are two principal circles, which must be real, and a node which is one of the points of intersection of these circles.

$$\text{(C.)} \qquad ax^2+2fyz=0,$$

the equation of the absolute being

$$x^2+y^2=4zw.$$

In this case there is only one principal circle $(x=0)$; the curve passes through the two common points of (x, y), and the point $(x=0, y=0, z=0)$ is a cusp on the curve.

It is also clear that circular cubics can be reduced to one of these three forms: since we have seen that the equation of the second degree represents a cubic when it is satisfied by the coordinates of the line at infinity.

CHAPTER VII.—CLASSIFICATION OF BICIRCULAR QUARTICS.

i. *Method.*—§ 85.

85. It will be convenient to take as the basis of our classification the nature of the roots of the discriminating quartic; we shall thus have three species, each of which may be subdivided into two—according as the double points at infinity are nodes or cusps. We shall then have three similar species of circular cubics.

Using the notation employed by SALMON ('Higher Plane Curves,' § 82), we shall denote the characteristics of a curve by $(m, n, \delta, \tau, \kappa, i)$, and we see that we shall have the following cases :—

	m	n	δ	τ	κ	i	Name.
i.	4	8	2	8	0	12	
ii.	4	6	0	1	2	8	Cartesian
iii.	4	6	3	4	0	6	
iv.	4	4	1	1	2	2	Limaçon
v.	4	5	2	2	1	4	
vi.	4	3	0	0	3	0	Cardioid
vii.	3	6	0	0	0	9	
viii.	3	4	1	0	0	3	
ix.	3	3	0	0	1	1	

(i.) may be called the general bicircular quartic; (iii.) is the general inverse of a conic; (v.) is the inverse of a parabola; (vii.) may be called the general circular cubic; (viii.) is the inverse of a conic with respect to a point on the curve; and (ix.) is the inverse of a parabola with respect to a point on the curve.

General Bicircular Quartic.—§§ 86–92.

86. The equation of the curve may be written

$$ax^2 + by^2 + cz^2 + dw^2 = 0;$$

and if we write the absolute

$$\psi \equiv x^2 + y^2 + z^2 + w^2 = 0,$$

the coordinates of the line at infinity will be $\dfrac{1}{r_1}, \dfrac{1}{r_2}, \dfrac{1}{r_3}, \dfrac{1}{r_4}$, *i.e.*, the reciprocals of the radii of the principal circles.

87. The coordinates of any tangent circle at the point $(x'y'z'w')$ will be proportional to

$$(a+k)x', \quad (b+k)y', \quad (c+k)z', \quad (d+k)w'.$$

The equation to the tangent at the point is, by equation (98),

$$(x'x+y'y+z'z+w'w)\left(\frac{ax'}{r_1}+\frac{by'}{r_2}+\frac{cz'}{r_3}+\frac{dw'}{r_4}\right)$$

$$=(ax'x+by'y+cz'z+dw'w)\left(\frac{x'}{r_1}+\frac{y'}{r_2}+\frac{z'}{r_3}+\frac{w'}{r_4}\right). \quad \ldots \quad (104)$$

The equation to the normal at the point $(x'y'z'w')$ is, by equation (101),

$$\begin{vmatrix} x, & y, & z, & w \\ x', & y', & z', & w' \\ ax', & by', & cz', & dw' \\ \dfrac{1}{r_1}, & \dfrac{1}{r_2}, & \dfrac{1}{r_3}, & \dfrac{1}{r_4} \end{vmatrix} = 0. \quad \ldots \quad \ldots \quad (105)$$

88. The coordinates of any bitangent circle being $(\xi, \eta, \zeta, 0)$ we must have

$$\frac{\xi}{(a-d)x'}=\frac{\eta}{(b-d)y'}=\frac{\zeta}{(c-d)z'}.$$

Hence we must have

$$\frac{\xi^2}{a-d}+\frac{\eta^2}{b-d}+\frac{\zeta^2}{c-d}=0 \quad \ldots \quad \ldots \quad \ldots \quad (106)$$

89. The pair of double tangents which belong to this system of bitangent circles are given by

$$\frac{\lambda^2}{a-d}+\frac{\mu^2}{b-d}+\frac{\nu^2}{c-d}=0,$$

where

$$\frac{\lambda}{r_1}+\frac{\mu}{r_2}+\frac{\nu}{r_3}=0.$$

If ϕ be the angle between them, we can deduce at once from § 65, remembering that

$$\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}+\frac{1}{r_4^2}=0 \; ;$$

$$\tan\phi=\frac{\dfrac{2}{r_4}\left\{\dfrac{1}{(a-d)(b-d)(c-d)}\left(\dfrac{a}{r_1^2}+\dfrac{b}{r_2^2}+\dfrac{c}{r_3^2}+\dfrac{d}{r_4^2}\right)\right\}^{\frac{1}{2}}}{\dfrac{1}{a-d}\left(\dfrac{1}{r_2^2}+\dfrac{1}{r_3^2}\right)+\dfrac{1}{b-d}\left(\dfrac{1}{r_3^2}+\dfrac{1}{r_1^2}\right)+\dfrac{1}{c-d}\left(\dfrac{1}{r_1^2}+\dfrac{1}{r_2^2}\right)} \quad \ldots \quad \ldots \quad (107)$$

90. Since the foci may be considered as bitangent circles whose radii are indefinitely

small, the coordinates of the four foci belonging to the system, given by equation (106), will be given by

$$\frac{x^2}{a-d}+\frac{y^2}{b-d}+\frac{z^2}{c-d}=0,$$

and

$$x^2+y^2+z^2=0 ;$$

whence

$$\frac{x^2}{(b-c)(a-d)}=\frac{y^2}{(c-a)(b-d)}=\frac{z^2}{(a-b)(c-d)}. \quad . \quad . \quad . \quad . \quad . \quad (108)$$

91. From the form of these equations it follows, that all curves whose equations are of the form

$$\frac{x^2}{\alpha^2+k}+\frac{y^2}{\beta^2+k}+\frac{z^2}{\gamma^2+k}+\frac{w^2}{\delta^2+k}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (109)$$

and

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}+\frac{w^2}{\delta^2}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (110)$$

will have the same foci.

Subtracting these equations, we have

$$\frac{x^2}{\alpha^2(\alpha^2+k)}+\frac{y^2}{\beta^2(\beta^2+k)}+\frac{z^2}{\gamma^2(\gamma^2+k)}+\frac{w^2}{\delta^2(\delta^2+k)}=0.$$

Hence the circles whose coordinates are respectively proportional to

$$\frac{x}{\alpha^2}, \quad \frac{y}{\beta^2}, \quad \frac{z}{\gamma^2}, \quad \frac{w}{\delta^2} ;$$

$$\frac{x}{\alpha^2+k}, \quad \frac{y}{\beta^2+k}, \quad \frac{z}{\gamma^2+k}, \quad \frac{w}{\delta^2+k} ;$$

must cut orthogonally; but these circles touch the curves given by (109), (110) at their common points; hence confocal bicircular quartics cut orthogonally.

Through any point, two quartics can be drawn confocal with a given bicircular quartic, since the equation (109) is a quadratic in k. We see, too, that two circular cubics can be drawn confocal with a given bicircular quartic.

92. Let $(\xi\eta\zeta\omega)$ be the coordinates of any circle S; this will cut orthogonally one of the bitangent circles, at the point $(x'y'z'w')$ on the curve

$$ax^2+by^2+cz^2+dw^2=0,$$

if

$$(a-d)x'\xi+(b-d)y'\eta+(c-d)z'\zeta=0.$$

It follows that two bitangent circles belonging to this system can be drawn to cut S orthogonally; and their four points of contact lie on the circle

$$(a-d)\xi x+(b-d)\eta y+(c-d)\zeta z=0.$$

This circle intersects S in points lying on the circle

$$a\xi x + b\eta y + c\zeta z + d\omega w = 0.$$

Hence, given any circle S, four pairs of bitangent circles can be drawn to any bicircular quartic, cutting S orthogonally; and their points of contact lie on four circles, which have with S a common radical axis.

ii. *Cartesian Oval.*—§§ 93, 94.

93. If one of the principal circles has its radius infinite, the curve will be symmetrical with respect to the axis, which will pass through the centres of the other three principal circles. If the foci which lie on this axis coincide with these centres, the curve must have cusps at the circular points at infinity. Let us suppose the circle, whose radius was r_4 in § 86, to become the axis; then by § 90, the coordinates of the foci on the axis will be

$$\frac{x^2}{(b-c)(a-d)} = \frac{y^2}{(c-a)(b-d)} = \frac{z^2}{(a-b)(c-d)}.$$

If these points are the centres of the principal circles we must have

$$r_1^2(b-c)(a-d) = r_2^2(c-a)(b-d) = r_3^2(a-b)(c-d); \quad . \quad . \quad . \quad (111)$$

which is equivalent to only one relation between the coefficients, viz.:—

$$\frac{1}{a-d} \cdot \frac{1}{r_1^2} + \frac{1}{b-d} \cdot \frac{1}{r_2^2} + \frac{1}{c-d} \cdot \frac{1}{r_3^2} = 0,$$

since

$$\frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} = 0.$$

Again the double tangents at right angles to the axis are given by

$$\frac{\lambda^2}{a-d} + \frac{\mu^2}{b-d} + \frac{\nu^2}{c-d} = 0,$$

$$\frac{\lambda}{r_1} + \frac{\mu}{r_2} + \frac{\nu}{r_3} = 0;$$

which are clearly satisfied by taking

$$\lambda r_1 = \mu r_2 = \nu r_3;$$

thus one of them coincides with the line at infinity, and so there is but one proper double tangent.

94. There are two finite foci on each principal circle, their coordinates being respectively proportional to

$$0, \quad \frac{1}{r_3}, \quad \frac{1}{r_2}, \quad \pm\frac{1}{r_1},$$

$$\frac{1}{r_3}, \quad 0, \quad \frac{1}{r_1}, \quad \pm\frac{1}{r_2},$$

$$\frac{1}{r_2}, \quad \frac{1}{r_1}, \quad 0, \quad \pm\frac{1}{r_3}.$$

But they are all imaginary.

iii. *Bicircular Quartics having a Third Node.*—§§ 95–102.

95. The equation of the curve may be reduced (by § 83) to the form

$$ax^2 + by^2 + cz^2 = 0,$$

the equation of the absolute being

$$x^2 + y^2 = 4zw \; ;$$

and if r_1, r_2 are the radii of the two principal circles, e the distance between their points of intersection, then the coordinates of the line at infinity are $\dfrac{1}{r_1}, \dfrac{1}{r_2}, \dfrac{1}{e}, \dfrac{1}{e}$.

96. The coordinates of any circle touching the curve at the point $(x'y'z'w')$ must be proportional to $(\xi, \eta, \zeta, \omega)$, where

$$\frac{\xi}{(a+k)x'} = \frac{\eta}{(b+k)y'} = \frac{-2\omega}{cz'-2kw'} = \frac{-2\zeta}{-2kz'}; \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (112)$$

and, by § 72, the equation to the tangent line at $(x'y'z'w')$ will be

$$\left(\frac{ax'}{r_1} + \frac{by'}{r_2} + \frac{cz'}{e}\right)(xx' + yy' - 2zw' - 2wz') = \left(\frac{x'}{r_1} + \frac{y'}{r_2} - 2\frac{w'+z'}{e}\right)(axx' + byy' + czz'); \; (113)$$

also the equation to the normal at the point $(x'y'z'w')$ will be, by equation (101),

$$\begin{vmatrix} x, & y, & -2w, & -2z \\ x', & y', & -2w', & -2z' \\ ax', & by', & cz', & 0 \\ \dfrac{1}{r_1}, & \dfrac{1}{r_2}, & -\dfrac{2}{e}, & -\dfrac{2}{e} \end{vmatrix} = 0. \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (114)$$

97. The circle $(\xi\eta\zeta\omega)$ given by (112) will be a bitangent circle, if $k = -a$ or $-b$. In the former case we have

$$\frac{\eta}{(b-a)y'} = \frac{-2\zeta}{+2az'} = \frac{-2\omega}{cz'+2aw'} = \frac{c\zeta - 2a\omega}{2a^2w'}.$$

Hence

$$\frac{\eta^2}{b-a}-\frac{c\zeta^2}{a^2}+\frac{4\omega\zeta}{a}=0. \qquad \qquad \qquad (115)$$

98. The double tangents which belong to this system of bitangent circles are given by

$$\frac{\mu^2}{b-a}-\frac{c\nu^2}{a^2}+\frac{4\nu\rho}{a}=0,$$

and

$$\frac{\mu}{r_2}-\frac{2\nu}{e}-\frac{2\rho}{e}=0.$$

If ϕ be the angle between them, we shall have, since

$$\frac{1}{r_1{}^2}+\frac{1}{r_2{}^2}=\frac{4}{e^2},$$

$$\tan\phi=\frac{\dfrac{2}{ar_1}\left\{\dfrac{1}{b-a}\left(\dfrac{a}{r_1{}^2}+\dfrac{b}{r_2{}^2}+\dfrac{c}{e^2}\right)\right\}^{\frac{1}{2}}}{\dfrac{2}{ar_2{}^2}-\dfrac{c}{a^2e^2}-\dfrac{4}{ae^2}+\dfrac{4}{b-a}\dfrac{1}{e^2}} \qquad \qquad \qquad (116)$$

Hence these double tangents cannot coincide unless

$$\frac{a}{r_1{}^2}+\frac{b}{r_2{}^2}+\frac{c}{e_2{}^2}=0,$$

in which case, the double tangents from the centre of the other principal circle coincide also; but this equation is the condition that the curve should be a cubic.

99. If in (112) we take $k=0$, we have a series of tangent circles passing through the node, two of these circles will reduce to straight lines, which will be the tangents from the node; in this case we shall have

$$\frac{\lambda}{ax'}=\frac{\mu}{by'}=\frac{\nu}{0}=\frac{-2\rho}{cz'},$$

whence

$$\frac{\lambda^2}{a}+\frac{\mu^2}{b}+\frac{4\rho^2}{c}=0\,;$$

where

$$\frac{\lambda}{r_1}+\frac{\mu}{r_2}-\frac{2\rho}{e}=0\,;$$

and the angle between them is given by

$$\tan\phi=\frac{\left\{\dfrac{-4e}{abc}\left(\dfrac{a}{r_1{}^2}+\dfrac{b}{r_2{}^2}+\dfrac{c}{e^2}\right)\right\}^{\frac{1}{2}}}{\dfrac{4}{c}+\dfrac{1}{a}+\dfrac{1}{b}}.$$

These also coincide if the curve is a cubic.

100. The foci corresponding to the principal circle $(x=0)$ are given by

$$\frac{y^2}{b-a}-\frac{cz^2}{a^2}+\frac{4wz}{a}=0,$$

and

$$y^2=4zw.$$

Hence we have

$$\frac{y^2}{c(b-a)}=\frac{z^2}{ab}=\frac{4zw}{c(b-a)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (117)$$

Similarly the coordinates of the two foci on the circle $y=0$, may be written down.

101. From the form of equations (117), it appears that all curves given by the equation

$$\frac{x^2}{a^2+\kappa}+\frac{y^2}{b^2+\kappa}+\frac{z^2}{c^2}=0,$$

are confocal with the curve

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0.$$

Subtracting, we have

$$\frac{x^2}{a^2(a^2+\kappa)}+\frac{y^2}{b^2(b^2+\kappa)}=0.$$

Hence the circles whose coordinates are respectively proportional to

$$\frac{x}{a^2}, \qquad \frac{y}{b^2}, \qquad 0, \qquad \frac{-z}{2c^2};$$

$$\frac{x}{a^2+\kappa}, \qquad \frac{y}{b^2+\kappa}, \qquad 0, \qquad \frac{-z}{2c^2};$$

must cut orthogonally, but these circles touch the curves at their common points; hence confocal curves cut orthogonally.

Since we have a quadratic to determine κ when (xyz) are given, through any point two curves can be drawn confocal with a given curve; and two nodal circular cubics can also be drawn with the same node and confocal with a given nodal bicircular quartic.

102. The equation

$$ax^2+by^2+cz^2=0$$

represents in general a nodal bicircular quartic, and by inverting with respect to the node, we see that it is the inverse curve of an ellipse or hyperbola, according as a and b have the same or opposite signs, with respect to some point in the same plane.

Such a curve then has two principal circles, with two single foci on each: it has also four double tangents, two from the centre of each principal circle.

If one of the principal circles becomes a straight line, it divides the curve

symmetrically ; corresponding, in fact, to the case where a conic is inverted on a point in one of its axes.

If the radii of both principal circles are infinite the centre of inversion is the centre of the conic : *e.g.*, the lemniscate, whose equation would be of the form $z^2 = a^2(x^2 + y^2)$.

<div align="center">iv. <i>The Limaçon.</i>—§ 103.</div>

103. If one of the principal circles, in the last section, becomes a straight line, and one of the two foci coincide with the centre of the principal circle, the nodes at infinity become cusps.   This case corresponds to inversion of a conic on a focus.

Suppose that r_2 is infinite in § 95, then the condition that the curve should be a Limaçon is, from equation (117),

$$4r_1^2 c(a-b) = 4e^2 ab = e^2 c(a-b) \ ;$$

which, since we must have

$$\frac{1}{r_1^2} = \frac{4}{e^2},$$

becomes

$$4ab = c(a-b). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (118)$$

The double tangents perpendicular to the axis are given by

$$\frac{\lambda^2}{a-b} - \frac{c\nu^2}{b^2} + \frac{4\nu\rho}{b} = 0,$$

and

$$\frac{\lambda}{r_1} - \frac{2\nu}{e} - \frac{2\rho}{e} = 0 \ ;$$

which equations are satisfied by $\lambda r_1 = \nu e = \rho e$: so that there is only one double tangent.

<div align="center">v. <i>Bicircular Quartic having a Cusp.</i>—§§ 104–109.</div>

104. The equation of the curve may be reduced (by § 83) to the form

$$x^2 = 2ayz.$$

The system of reference being the principal circle $(x=0)$; the circle $(y=0)$ passing through the cusp, and the other point in which the curve cuts the principal circle ; the cusp $(z=0)$; and the other point $(w=0)$ common to the curve and principal circle.

Let r_1, r_2 be the radii of the two circles, e the distance between their points of

intersection; $\dfrac{1}{r_1}, \dfrac{1}{r_2}, \dfrac{1}{e}, \dfrac{1}{e}$ the coordinates of the line at infinity; then the equation to the absolute will be

$$x^2 + y^2 = 4zw.$$

The curve is clearly the inverse of a parabola.

105. If $(\xi \eta \zeta \omega)$ be the coordinates of any circle touching the curve at the point $(x'y'z'w')$, we must have

$$\frac{\xi}{(1+k)x'} = \frac{\eta}{-az'+ky'} = \frac{-2\omega}{-ay'-2kw'} = \frac{-2\zeta}{-2kz'}. \quad \cdots \cdots \quad (119)$$

The equation to the tangent line at the point $(x'y'z'w')$ will be, by § 72,

$$\left(\frac{1}{r_1}x' - \frac{a}{r_2}z' - \frac{a}{e}y'\right)(xx' + yy' - 2zw' - 2wz')$$

$$= \left(\frac{1}{r_1}x' + \frac{1}{r_2}y' - 2\frac{z'+w'}{e}\right)(xx' - az'y - azy'); \quad \cdots \cdots \quad (120)$$

the equation to the normal being

$$\begin{vmatrix} x, & y, & -2w, & -2z \\ x', & y', & -2w', & -2z' \\ x', & -az', & -ay', & 0 \\ \dfrac{1}{r_1}, & \dfrac{1}{r_2}, & -\dfrac{2}{e}, & -\dfrac{2}{e} \end{vmatrix} = 0. \quad \cdots \cdots \quad (121)$$

106. The circle $(\xi \eta \zeta \omega)$ will be a bitangent circle, if $k = -1$, in which case

$$\frac{\eta}{az'+y'} = \frac{\zeta}{z'} = \frac{2\omega}{2w'-ay'} = \frac{\eta - a\zeta}{y'} = \frac{2\omega + a\eta - a^2\zeta}{2w'};$$

whence

$$(\eta - a\zeta)^2 = 4\zeta\omega. \quad \cdots \cdots \cdots \quad (122)$$

107. The two double tangents are given by

$$\mu^2 - 2a\mu\nu + a^2\nu^2 - 4\nu\rho = 0,$$

$$\frac{\mu}{r_2} - \frac{2\nu}{e} - \frac{2\rho}{e} = 0;$$

the angle ϕ between them being given by

$$\tan\phi = \frac{\dfrac{2}{r_1}\left\{\left(\dfrac{2a}{er_2} - \dfrac{1}{r_1^2}\right)\right\}^{\frac{1}{2}}}{\dfrac{a^2 + 8}{e^2} - \dfrac{2a}{er_2} - \dfrac{2}{r_2^2}}. \quad \cdots \cdots \cdots \quad (123)$$

These can only coincide if

$$\frac{2a}{er_2} = \frac{1}{r_1^2} \; ;$$

in which case the curve is a circular cubic.

108. Taking $k=0$ in (119), we have a series of tangent circles passing through the cusp; to find the tangents from the cusp to the curve we shall have

$$\frac{\lambda}{x'} = \frac{\mu}{-az'} = \frac{-2\rho}{-ay'},$$

whence

$$a\lambda^2 + 4\mu\rho = 0 \; ;$$

where

$$\frac{\lambda}{r_1} + \frac{\mu}{r_2} - \frac{2\rho}{e} = 0.$$

The angle between these is given by

$$\tan\phi = \frac{2\left\{\dfrac{1}{r_1^2} - \dfrac{2a}{er_2}\right\}^{\frac{1}{2}}}{\dfrac{a}{e} + \dfrac{2}{r_2}}.$$

109. The focus of the curve is given by

$$(y-az)^2 = 4wz = y^2,$$

whence

$$\frac{y}{a} = \frac{z}{2} = \frac{8w}{a^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (124)$$

vi. *The Cardioid.*—§ 110.

110. If the radius r_1 of the principal circle of the curve

$$x^2 = 2ayz$$

become infinite, the curve is symmetrical with respect to the axis; and if the single focus on this axis is at an infinite distance, the curve has cusps at infinity, and is called the cardioid; being the inverse of a parabola with respect to its focus.

Referring to equation (124), we see that the condition is

$$a = 4 \; ;$$

so that the equation to a cardioid is

$$x^2 = 8yz. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (125)$$

vii. *General Circular Cubic.*—§§ 111, 112.

111. We have already seen that the equation

$$ax^2 + by^2 + cz^2 + dw^2 = 0,$$

represents a circular cubic when

$$\frac{a}{r_1{}^2} + \frac{b}{r_2{}^2} + \frac{c}{r_3{}^2} + \frac{d}{r_4{}^2} = 0, \quad \ldots \ldots \ldots \ldots \quad (126)$$

where r_1, r_2, r_3, r_4 are the radii of the four principal circles. The curve also passes through the centre of each of these circles.

By equation (104) we see that the equation to the asymptote (*i.e.*, tangent at the point $\frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r_3}, \frac{1}{r_4}$) is

$$\frac{ax}{r_1} + \frac{by}{r_2} + \frac{cz}{r_3} + \frac{dw}{r} = 0 \; ; \quad \ldots \ldots \ldots \ldots \quad (127)$$

the equation to the tangent at the centre of the circle ($x = 0$) being

$$\frac{a-b}{r_2} y + \frac{a-c}{r_3} z + \frac{a-d}{r_4} w = 0,$$

which is clearly parallel to the asymptote.

Hence the tangents to the curve at the centres of the principal circles are all parallel to the asymptote.

112. As in the case of the general bicircular quartic, there will be four systems of bitangent circles, and on each principal circle there will be four single foci. There are clearly no double tangents. And if one of the principal circles degenerates into a straight line, the asymptote is perpendicular to it.

viii. *Nodal Circular Cubic.*—§ 113.

113. The equation considered in § 95,

$$ax^2 + by^2 + cz^2 = 0,$$

is a circular cubic, when

$$\frac{a}{r_1{}^2} + \frac{b}{r_2{}^2} + \frac{c}{e^2} = 0.$$

This curve is the inverse of a conic with respect to a point on the curve. The curve passes through the centres of the principal circles, the tangents being respectively, by equation (113),

$$(a-b)\frac{y}{r_2} - \frac{2a+c}{e}z - 2\frac{a}{e}w = 0,$$

$$\frac{(b-a)}{r_1}x - \frac{2b+c}{e}z - 2\frac{b}{e}w = 0\;;$$

which are parallel to the straight line

$$\frac{ax}{r_1} + \frac{by}{r_2} + \frac{cz}{e} = 0,$$

which is the line joining the node to the point in which the line at infinity cuts the curve, so that these three lines are parallel to the asymptote.

As in the case of the nodal bicircular quartic there will be two single foci on each principal circle, and two corresponding systems of bitangent circles.

ix. *Cuspidal Circular Cubic.*—§ 114.

114. The equation

$$x^2 = 2ayz$$

represents a circular cubic, having the point $z=0$ for a cusp, when

$$\frac{1}{r_1^2} = \frac{2a}{er_2}\;;$$

r_1 being the radius of its principal circle, r_2 that of a circle cutting this orthogonally, and passing through the cusp, and the other point common to the curve and its principal circle.

The curve clearly passes through the centre of its principal circle, the tangent at the point being

$$y\left(\frac{1}{r_2} + \frac{a}{e}\right) - \frac{ez}{2r_2^2} - \frac{2w}{e} = 0,$$

which is parallel to

$$\frac{x}{r_1} - \frac{ay}{e} - \frac{az}{r_2} = 0,$$

the line joining the cusp to the third point in which the curve cuts the line at infinity; hence these are parallel to the asymptote.

The curve has one system of bitangent circles, and one focus which lies on the principal circle.

CHAPTER VIII.—MISCELLANEOUS THEOREMS.

Equation of an Anallagmatic Curve referred to Three Circles Orthogonal to the same Principal Circle.—§§ 115–121.

115. If the system of circles (1, 2, 3, 4) be such that (4) is orthogonal to (1, 2, 3), the equation of the absolute must be of the form

$$w^2 + f(xyz) = 0 ;$$

also if (4) be a principal circle of an anallagmatic curve, its equation must be of the same form; by subtraction we see that the equation,

$$ax^2 + by^2 + cz^2 + 2fgz + 2gzx + 2hxy = 0, \quad \ldots \quad \ldots \quad (128)$$

may be considered as the general equation of such a curve referred to any three circles cutting one of its principal circles orthogonally.

Thus, for any theorem proved in the case of conics we can easily derive an analogous theorem for bicircular quartics or circular cubics.

116. Any bitangent circle of the system which cuts the given principal circle orthogonally, must have for its equation

$$\alpha x + \beta y + \gamma z = 0, \quad \ldots \quad \ldots \quad \ldots \quad (129)$$

and since it touches (128), we shall have

$$\begin{vmatrix} a, & h, & g, & \alpha \\ h, & b, & f, & \beta \\ g, & f, & c, & \gamma \\ \alpha, & \beta, & \gamma, & 0 \end{vmatrix} = 0. \quad \ldots \quad \ldots \quad (130)$$

Referring to § 24, equation (22), we see that α, β, γ are proportional to the areal coordinates of the centre of the circle (129) referred to the triangle formed by joining the centres of the circles (1, 2, 3), provided that x, y, z are proportional to the powers of a point with respect to these circles.

We see, then, by equation (130), that the locus of the centres of all bitangent circles of the same system is a conic; which is called by Dr. CASEY the focal conic of the system.

117. Suppose now the circles (1, 2, 3) to be the other principal circles, then the equation to the curve must be of the form

$$ax^2 + by^2 + cz^2 = 0,$$

and the corresponding focal conic is

$$\frac{\alpha^2}{a}+\frac{\beta^2}{b}+\frac{\gamma^2}{c}=0.$$

And we see that the focal conic corresponding to one principal circle is self-conjugate with respect to the triangle formed by the centres of the other three.

118. Or, again, in the case of a nodal curve; let $(1, 2, 3)$ denote the other principal circle, and its two points of intersection with (4); the equation of the curve is of the form

$$ax^2+by^2+2fyz=0;$$

and the corresponding focal conic

$$f^2\alpha^2-ab\gamma^2+2af\beta\gamma=0,$$

which also passes through the node.

119. If the system of reference be three bitangent circles, the equation to the curve must be of the form

$$a\sqrt{x}+b\sqrt{y}+c\sqrt{z}=0 ; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (131)$$

and in that case the focal conic is

$$\frac{a^2}{\alpha}+\frac{b^2}{\beta}+\frac{c^2}{\gamma}=0.$$

In particular we see that if A, B, C be three foci on the same principal circle of the quartic, or cubic, and P any point on the curve, we must have

$$a.\mathrm{AP}+b.\mathrm{BP}+c.\mathrm{CP}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (132)$$

120. Suppose that the curve is a Cartesian, having cusps at infinity, then the focal conics become circles.

It follows that equation (131) will represent a Cartesian if a, b, c are proportional to the sides of the triangle formed by the centres of the three bitangent circles (xyz). Thus we have the theorem that, the sum of the products of the tangents, from any point on a Cartesian to any three bitangent circles of the same system, into the corresponding sides of the triangle, formed by the centres of the circles, is zero.

121. Let the circles $(2, 3)$ be any two bitangent circles, and let (1) be the circle passing through their four points of contact with the curve; then the equation of the quartic must take the form

$$x^2=2fyz,$$

and then the focal conic is

$$f\alpha^2=2\beta\gamma.$$

If (y, z) be foci, (x) might be called their directrix; and we see that the product of the distances of any point on a bicircular quartic from two foci on the same principal circle, is proportional to the square of the tangent from the point to their directrix.

Again, if (yz) denote the double tangents from the centre of the principal circle of the system, we see that the centre of the focal conic coincides with the centre of the polar circle of the centre of the principal circle, and the asymptotes of the focal conic are perpendicular to the double tangents. It follows, then, that the focal conic is an hyperbola or ellipse, according as these tangents are real or imaginary.

It follows, also, that the focal conic of a circular cubic is a parabola, whose axis is perpendicular to the asymptote of the cubic.

Circle of Curvature at any Point of an Anallagmatic Curve.—§§ 122–124.

122. Let the equation to the circle of curvature at the point $(x'y'z'w')$ be

$$\xi x + \eta y + \zeta z + \omega w = 0.$$

Then we must have ξ, η, ζ, ω, proportional to the minors of x, y, z, w, in the determinant

$$\begin{vmatrix} x, & y, & z, & w \\ x', & y', & z', & w' \\ \delta x', & \delta y', & \delta z', & \delta w' \\ \delta^2 x', & \delta^2 y', & \delta^2 z', & \delta^2 w' \end{vmatrix}.$$

And as we are merely concerned with the ratios of the coordinates (x, y, z, w), we may take w constant; so we shall have

$$\frac{\xi}{\delta y' \delta^2 z' - \delta z' \delta^2 y'} = \frac{\eta}{\delta z' \delta^2 x' - \delta x' \delta^2 z'} = \frac{\zeta}{\delta x' \delta^2 y' - \delta y' \delta^2 x'}.$$

123. If the equation of the curve referred to its principal circles be

$$ax^2 + by^2 + cz^2 + dw^2 = 0,$$

where the equation to the absolute is

$$x^2 + y^2 + z^2 + w^2 = 0;$$

we shall have

$$ax\delta x + by\delta y + cz\delta z = 0,$$
$$x\delta x + y\delta y + z\delta z = 0;$$

whence

$$\frac{x\delta x}{b-c} = \frac{y\delta y}{c-a} = \frac{z\delta z}{a-b} = u \text{ say};$$

hence

$$\delta^2 x = (b-c)\left(\frac{\delta u}{x} - \frac{u\delta x}{x^2}\right),$$

$$\delta^2 y = (c-a)\left(\frac{\delta u}{y} - \frac{u\delta y}{y^2}\right),$$

$$\delta^2 z = (a-b)\left(\frac{\delta u}{z} - \frac{u\delta z}{z^2}\right);$$

so that

$$\xi = (c-a)(a-b)\frac{u^2}{y'z'}\left(\frac{\delta z'}{z'} - \frac{\delta y'}{y'}\right)$$

$$= (c-a)(a-b)\frac{u^3}{y'z'}\left(\frac{a-b}{z'^2} - \frac{c-a}{y'^2}\right);$$

or since

$$(a-b)y'^2 + (a-c)z'^2 + (a-d)w'^2 = 0,$$

$$\xi = (a-b)(a-c)\frac{u^3}{y'^3 z'^3}(d-a)w'^2.$$

Hence

$$\frac{\xi}{(a-b)(a-c)(a-d)x'^3} = \frac{\eta}{(b-a)(b-c)(b-d)y'^3} = \frac{\zeta}{(c-a)(c-b)(c-d)z'^3} = \frac{\omega}{(d-a)(d-b)(d-c)w'^3}.$$

So that the equation of the circle of curvature at the point $(x'y'z'w')$ on the curve

$$ax^2 + by^2 + cz^2 + dw^2 = 0,$$

is

$$(a-b)(a-c)(a-d)x'^3 x + (b-a)(b-c)(b-d)y'^3 y$$
$$+ (c-a)(c-b)(c-d)z'^3 z + (d-a)(d-b)(d-c)w'^3 w = 0 ; \quad . \quad . \quad (133)$$

and the points of inflexion of the curve lie on the tricircular sextic,

$$\frac{(a-b)(a-c)(a-d)}{r_1}x^3 + \frac{(b-a)(b-c)(b-d)}{r_2}y^3 + \frac{(c-a)(c-b)(c-d)}{r_3}z^3$$

$$+ \frac{(d-a)(d-b)(d-c)}{r_4}w^3 = 0 . \quad . \quad . \quad . \quad . \quad . \quad . \quad (134)$$

124. If R be the radius of curvature at the point $(x'y'z'w')$ of the curve

$$ax^2 + by^2 + cz^2 + dw^2 = 0,$$

we shall have

$$R^2 = \frac{\xi^2 + \eta^2 + \zeta^2 + \omega^2}{\left(\dfrac{\xi}{r_1} + \dfrac{\eta}{r_2} + \dfrac{\zeta}{r_3} + \dfrac{\omega}{r_4}\right)^2}.$$

And it may be easily verified that

$$\xi^2+\eta^2+\zeta^2+\omega^2=(a^2x'^2+b^2y'^2+c^2z'^2+d^2w'^2)^3 \, ;$$

so that

$$\frac{(a^2x'^2+b^2y'^2+c^2z'^2+d^2w'^2)^{\frac{3}{2}}}{R}$$

$$=\frac{(a-b)(a-c)(a-d)}{r_1}x'^3+\frac{(b-a)(b-c)(b-d)}{r_2}y'^3+\frac{(c-a)(c-b)(c-d)}{r_3}z'^3$$

$$+\frac{(d-a)(d-b)(d-c)}{r_3}w'^3. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (135)$$

PART II.—SYSTEMS OF CIRCLES ON THE SURFACE OF A SPHERE.

CHAPTER I.—GENERAL SYSTEMS OF CIRCLES.

The Equation of a Small Circle on a Sphere.—§§ 125, 126.

125. Let ABC be a spherical triangle, having all its angles right angles : then, if we denote the sines of the perpendiculars from any point P on the sides of the triangle by x, y, z, we have at once

$$x^2+y^2+z^2=1 \, ;$$

and again, if (xyz), $(x'y'z')$ be any two points, ϕ the angular distance between them,

$$xx'+yy'+zz'= \cos \phi.$$

So that the equation of a small circle is of the form

$$ax+by+cz=1,$$

the coordinates of its centre, and its radius, being given by

$$\frac{\bar{x}}{a}=\frac{\bar{y}}{b}=\frac{\bar{z}}{c}=\frac{\cos r}{1} \, ;$$

and if (xyz) be any point, whose angular distance from the centre of the circle is ϕ, we have

$$ax+by+cz= \cos \phi \sec r.$$

It follows from this that, if the angle of intersection of the circles

$$ax+by+cz=1,$$
$$a'x+b'y+c'z=1,$$

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be ω, and the distance between the centres ϕ, then

$$\tan r \tan r' \cos \omega = 1 - \cos \phi \sec r \sec r'$$
$$= 1 - aa' - bb' - cc' \ . \quad . \quad . \quad . \quad . \quad . \quad (136)$$

126   The equation of a great circle will be of the form

$$ax + by + cz = 0 \ ;$$

and the angle of intersection of this with the circle

$$a'x + b'y + c z = 1,$$

will be given by

$$\tan r' \cos \omega = - \cos \phi \sec r'$$
$$= -(aa' + bb' + cc') \ ; \quad . \quad . \quad . \quad . \quad . \quad . \quad (137)$$

and for two great circles we shall have

$$\cos \omega = -(aa' + bb' + cc').$$

The Power of Two Circles on a Sphere.—§§ 127, 128.

127. If r, r' be the angular radii of two small circles on a sphere, ϕ the angular distance between their poles, and ω their angle of intersection, then either of the expressions

$$\tan r \tan r' \cos \omega, \quad 1 - \cos \phi \sec r \sec r',$$

may be defined as the power of the circles.   Denoting the power by π (or if S, S′ are any circles by $\pi_{\mathrm{S,S'}}$), we see that $\pi_{\mathrm{S,S'}} = 0$ if the circles cut at right angles, and if they touch $\pi_{\mathrm{S,S'}}^2 = \pi_{\mathrm{S,S}} \cdot \pi_{\mathrm{S',S'}}$.   In the case of a pair of points P, P′: $\pi_{\mathrm{P,P'}} = 1 - \cos \phi = 2 \sin^2 \frac{1}{2} \phi$, where ϕ is the distance between them.

It will be convenient to define the power of a great circle, with respect to a small circle, as the product of the tangent of the radius of the small circle and the cosine of the angle of intersection : thus

$$\pi = \tan r \cos \omega = - \cos \phi \sec r \ ;$$

and the power of two great circles, as the cosine of the angle between them.

128. If O be the pole of a small circle, radius r, and P any point on the sphere, then if Q be taken in the arc OP so that

$$\tan \tfrac{1}{2} OP \cdot \tan \tfrac{1}{2} OQ = \tan^2 \tfrac{1}{2} r,$$

Q may be called the inverse point of P with respect to the circle whose radius is r, or simply the inverse of P with respect to O.

It may be easily shown, if S, S′ be any two small circles, by forming the equations of the inverse circles with respect to A ($\S\,125$), that the expression

$$\frac{\pi_{S,\,S'}}{\sqrt{\pi_{0,\,S}\cdot\pi_{0,\,S'}}}$$

is invariable.

<center>General Theorems.—$\S\S\,129$–132.</center>

129. If $(1, 2, 3, 4, 5)$, $(6, 7, 8, 9, 10)$ denote any two systems of circles on the surface of a sphere, the powers of the former are connected with those of the latter by the identical relation

$$\Pi\begin{pmatrix}1,\,2,\,3,\,4,\,5\\6,\,7,\,8,\,9,\,10\end{pmatrix}=0.$$

This is at once proved by multiplying together the matrices,

$$\begin{vmatrix}1, & a_1, & b_1, & c_1\\1, & a_2, & b_2, & c_2\\1, & a_3, & b_3, & c_3\\1, & a_4, & b_4, & c_4\\1, & a_5, & b_5, & c_5\end{vmatrix},\qquad\begin{vmatrix}1, & -a_6, & -b_6, & -c_6\\1, & -a_7, & -b_7, & -c_7\\1, & -a_8, & -b_8, & -c_8\\1, & -a_9, & -b_9, & -c_9\\1, & -a_{10}, & -b_{10}, & -c_{10}\end{vmatrix}.$$

Whence we get

$$\begin{vmatrix}\pi_{1,6}, & \pi_{1,7}, & \pi_{1,8}, & \pi_{1,9}, & \pi_{1,10}\\\pi_{2,6}, & \pi_{2,7}, & \pi_{2,8}, & \pi_{2,9}, & \pi_{2,10}\\\pi_{3,6}, & \pi_{3,7}, & \pi_{3,8}, & \pi_{3,9}, & \pi_{3,10}\\\pi_{4,6}, & \pi_{4,7}, & \pi_{4,8}, & \pi_{4,9}, & \pi_{4,10}\\\pi_{5,6}, & \pi_{5,7}, & \pi_{5,8}, & \pi_{5,9}, & \pi_{5,10}\end{vmatrix}=0,$$

i.e.

$$\Pi\begin{pmatrix}1,\,2,\,3,\,4,\,5\\6,\,7,\,8,\,9,\,10\end{pmatrix}=0.\quad\cdots\cdots\cdots\quad(138)$$

130. It is evident that this result is true if one, or more, of the circles are great circles, provided that we interpreted the meaning of the symbol π in accordance with the definition in $\S\,127$. Again, it is true, if the radius of any of the circles is zero. And we also see it is true if any circle of either system is such that the coordinates of its centre are zero; *i.e.*, any circles of either system may be replaced by the

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imaginary circle at infinity, which we will denote by θ; and we see that we must take $\pi_{\theta,\theta}=1$, and

$$\pi_{S,\theta}=1 \text{ if S be any small circle,}$$

$$\pi_{P,\theta}=1 \text{ if P be any point,}$$

$$\pi_{S,\theta}=0 \text{ if S be any great circle.}$$

131. If then $\omega_{S,S'}$ be the angle of intersection of the circles S, S' we may deduce at once from equation (138):—

$$\begin{vmatrix} 1, & \cot r_5, & \cot r_6, & \cot r_7, & \cot r_8 \\ \cot r_1, & \cos \omega_{1,5}, & \cos \omega_{1,6}, & \cos \omega_{1,7}, & \cos \omega_{1,8} \\ \cot r_2, & \cos \omega_{2,5}, & \cos \omega_{2,6}, & \cos \omega_{2,7}, & \cos \omega_{2,8} \\ \cot r_3, & \cos \omega_{3,5}, & \cos \omega_{3,6}, & \cos \omega_{3,7}, & \cos \omega_{3,8} \\ \cot r_4, & \cos \omega_{4,5}, & \cos \omega_{4,6}, & \cos \omega_{4,7}, & \cos \omega_{4,8} \end{vmatrix} = 0. \quad . \quad . \quad . \quad (139)$$

132. Exactly as in § 8, we can prove that

$$\left\{ \Pi\begin{pmatrix} 1, 2, 3, 4 \\ 5, 6, 7, 8 \end{pmatrix} \right\}^2 = \Pi\begin{pmatrix} 1, 2, 3, 4 \\ 1, 2, 3, 4 \end{pmatrix} \times \Pi\begin{pmatrix} 5, 6, 7, 8 \\ 5, 6, 7, 8 \end{pmatrix}. \quad . \quad . \quad . \quad . \quad (140)$$

CHAPTER II.—SPECIAL SYSTEMS OF CIRCLES.

Circle Cutting Three Circles Orthogonally.—§§ 133, 134.

133. Let the circle cutting the system (1, 2, 3) be denoted by (x), then since

$$\Pi\begin{pmatrix} \theta, x, 1, 2, 3 \\ \theta, x, 1, 2, 3 \end{pmatrix} = 0,$$

we have

$$\pi_{x,x}.\Pi\begin{pmatrix} \theta, 1, 2, 3 \\ \theta, 1, 2, 3 \end{pmatrix} = \Pi\begin{pmatrix} 1, 2, 3 \\ 1, 2, 3 \end{pmatrix};$$

and if the equations of the circles (1, 2, 3) be of the form

$$a_1 x + b_1 y + c_1 z = 1,$$

we have at once

$$\Pi\begin{pmatrix} \theta, 1, 2, 3 \\ \theta, 1, 2, 3 \end{pmatrix} = \begin{vmatrix} 1, & 0, & 0, & 0 \\ 1, & -a_1, & -b_1, & -c_1 \\ 1, & -a_2, & -b_2, & -c_2 \\ 1, & -a_3, & -b_3, & -c_3 \end{vmatrix} \times \begin{vmatrix} 1, & 0, & 0, & 0 \\ 1, & a_1, & b_1, & c_1 \\ 1, & a_2, & b_2, & c_2 \\ 1, & a_3, & b_3, & c_3 \end{vmatrix}$$

$$= -\frac{36}{R^6} \sec^2 r_1 \sec^2 r_2 \sec^2 r_3 . \{V(1, 2, 3)\}^2; \quad . \quad . \quad . \quad . \quad (141)$$

where R is the radius of the sphere, and V(1, 2, 3) denotes the volume of the tetra-hedron formed by the centre of the sphere and the poles of the circles (1, 2, 3).

Hence the radius of the orthogonal circle of the system (1, 2, 3) is given by

$$\tan^2 r = \frac{\sin^2 r_1 \sin^2 r_2 \sin^2 r_3}{36} \frac{R^6}{\{V(1, 2, 3)\}^2} \times \begin{vmatrix} -1, & \cos \omega_{1,2}, & \cos \omega_{1,3} \\ \cos \omega_{2,1}, & -1, & \cos \omega_{2,3} \\ \cos \omega_{3,1}, & \cos \omega_{3,2}, & -1 \end{vmatrix}. \quad (142)$$

134. If the three circles meet in a point, r must be zero, hence

$$\Pi\binom{1,\, 2,\, 3}{1,\, 2,\, 3} = 0.$$

Four Circles having a Common or Orthogonal Circle.—§§ 135, 136.

135. Let (x) denote the common orthogonal circle of the system (1, 2, 3, 4), then, since

$$\Pi\binom{x,\, 1,\, 2,\, 3,\, 4}{y,\, 1,\, 2,\, 3,\, 4} = 0,$$

it follows that we must have

$$\Pi\binom{1,\, 2,\, 3,\, 4}{1,\, 2,\, 3,\, 4} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (143)$$

This is clearly the necessary and sufficient condition that the system (1, 2, 3, 4) may have a common orthogonal circle.

136. If (5, 6, 7, 8) be any other system of circles, we must also have

$$\Pi\binom{1,\, 2,\, 3,\, 4}{5,\, 6,\, 7,\, 8} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (144)$$

As a particular case, we have

$$\Pi\binom{1,\, 2,\, 3,\, 4}{x,\, 1,\, 2,\, 3} = 0,$$

where (x) denotes any other circle.

We deduce that

$$\pi_{x,1}.\Pi\binom{2,\, 3,\, 4}{1,\, 2,\, 3} + \pi_{x,2}.\Pi\binom{1,\, 4,\, 3}{1,\, 2,\, 3} + \pi_{x,3}.\Pi\binom{1,\, 2,\, 4}{1,\, 2,\, 3} = \pi_{x,4}.\Pi\binom{1,\, 2,\, 3}{1,\, 2,\, 3} \quad . \quad . \quad (145)$$

But from (144) we can deduce, as in § 24, that,

$$\left\{\Pi\binom{1,\, 2,\, 3}{5,\, 6,\, 7}\right\}^2 = \Pi\binom{1,\, 2,\, 3}{1,\, 2,\, 3} \times \Pi\binom{5,\, 6,\, 7}{5,\, 6,\, 7};$$

so that (145) may be written,

$$\pi_{x,4}\cdot\left\{\Pi\begin{pmatrix}1,\,2,\,3\\1,\,2,\,3\end{pmatrix}\right\}^{\frac{1}{2}}=\pi_{x,1}\left\{\Pi\begin{pmatrix}2,\,3,\,4\\2,\,3,\,4\end{pmatrix}\right\}^{\frac{1}{2}}+\pi_{x,2}\left\{\Pi\begin{pmatrix}1,\,4,\,3\\1,\,4,\,3\end{pmatrix}\right\}^{\frac{1}{2}}+\pi_{x,3}\left\{\Pi\begin{pmatrix}1,\,2,\,4\\1,\,2,\,4\end{pmatrix}\right\}^{\frac{1}{2}}.$$

But if r be the radius of the common orthogonal circle, we have, by § 133,

$$\frac{36}{R^6}\tan^2 r^2=\cos^2 r_1\cos^2 r_2\cos^2 r_3\frac{\Pi\begin{pmatrix}1,\,2,\,3\\1,\,2,\,3\end{pmatrix}}{\{V(1,2,3)\}^2}=\cos^2 r_2\cos^2 r_3\cos^2 r_4\frac{\Pi\begin{pmatrix}2,\,3,\,4\\2,\,3,\,4\end{pmatrix}}{\{V(2,3,4)\}^2}=\&c.\ .$$

Hence we have

$$\pi_{x,4}\cdot\cos r_4.V(1,2,3)=\pi_{x,1}\cdot\cos r_1.V(2,3,4)+\pi_{x,2}\cdot\cos r_2.V(1,4,3)$$
$$+\pi_{x,3}\cdot\cos r_3.V(1,2,4)\ ;\quad .\quad(146)$$

which result may also be written

$$\pi_{x,4}\cdot\cos r_4=\alpha.\cos r_1.\pi_{x,1}+\beta.\cos r_2.\pi_{x,2}+\gamma.\cos r_3.\pi_{x,3}\ ;\quad.\quad.\quad.\quad(147)$$

where α, β, γ may be defined as the areal coordinates of the pole of the circle (4) with respect to the triangle formed by the poles of the circles (1, 2, 3).

Thus, if A, B, C be the triangle, P the pole of (4), then

$$\alpha=\frac{V(P,\,B,\,C)}{V(A,\,B,\,C)}=\frac{\sin\,(\text{perp. from P on BC})}{\sin\,(\text{perp. from A on BC})}.$$

As a particular case of (147), let x be a point, O say, then A, B, C being the centres of (1, 2, 3), and P being a point on the circle which cuts them orthogonally, we shall have

$$1-\cos OP=\alpha(\cos r_1-\cos OA)+\beta(\cos r_2-\cos OB)+\gamma(\cos r_3-\cos OC),$$

or more generally, P being the pole of a circle which, with (1, 2, 3), has a common orthogonal circle,

$$\cos r_4-\cos OP=\alpha(\cos r_1-\cos OA)+\beta(\cos r_2-\cos OB)+\gamma(\cos r_3-\cos OC).$$

Orthogonal Systems.—§§ 137–139.

137. Four circles may be said to form an orthogonal system if each one cuts the other three orthogonally. It is clear that the pole of any one of four such circles must be the orthocentre of triangle formed by the poles of the other three.

Let $(1, 2, 3, 4)$ be such a system, then if (x, y) be any other circles we must have from the equation

$$\Pi\begin{pmatrix} x, 1, 2, 3, 4 \\ y, 1, 2, 3, 4 \end{pmatrix} = 0,$$

$$\pi_{x,y} = \frac{\pi_{x,1} . \pi_{y,1}}{\pi_{1,1}} + \frac{\pi_{x,2} . \pi_{y,2}}{\pi_{2,2}} + \frac{\pi_{x,3} . \pi_{y,3}}{\pi_{3,3}} + \frac{\pi_{x,4} . \pi_{y,4}}{\pi_{4,4}} \; ; \qquad \cdots \quad (148)$$

whence we have as particular cases

(i.) $\pi_{x,1} \cot^2 r_1 + \pi_{x,2} \cot^2 r_2 + \pi_{x,3} \cot^2 r_3 + \pi_{x,4} \cot^2 r_4 = -1,$

(ii.) $\pi_{x,1}^2 \cot^2 r_1 + \pi_{x,2}^2 \cot^2 r_2 + \pi_{x,3}^2 \cot^2 r_3 + \pi_{x,4}^2 \cot^2 r_4 = \tan^2 r_x,$

where x denotes any circle, radius r_x ;

(iii.) $\pi_{z,1} \cot^2 r_1 + \pi_{z,2} \cot^2 r_2 + \pi_{z,3} \cot^2 r_3 + \pi_{z,4} \cot^2 r_4 = 0,$

(iv.) $\pi_{z,1}^2 \cot^2 r_1 + \pi_{z,2}^2 \cot^2 r_2 + \pi_{z,3}^2 \cot^2 r_3 + \pi_{z,4}^2 \cot^2 r_4 = 1,$

where z denotes any great circle ;

(v.) $\cot^2 r_1 + \cot^2 r_2 + \cot^2 r_3 + \cot^2 r_4 = -1 \; ; \qquad \cdots \quad (149)$

so that one of the circles must be imaginary.

138. If the circles $(1, 2, 3, 4)$ form a system not having a common orthogonal circle, we may find four other circles, $(5, 6, 7, 8)$ say, such that each of the latter is orthogonal to three of the former. One such system may be called the "orthogonal system" of the other.

Let x, y denote any two circles, then since

$$\Pi\begin{pmatrix} x, 1, 2, 3, 4 \\ y, 5, 6, 7, 8 \end{pmatrix} = 0,$$

we shall have

$$\pi_{x,y} = \frac{\pi_{x,5} . \pi_{y,1}}{\pi_{1,5}} + \frac{\pi_{x,6} . \pi_{y,2}}{\pi_{2,6}} + \frac{\pi_{x,7} . \pi_{y,3}}{\pi_{3,7}} + \frac{\pi_{x,8} . \pi_{y,4}}{\pi_{4,8}} \; ; \qquad \cdots \quad (150)$$

whence we obtain as particular cases

$$\frac{1}{\pi_{1,5}} + \frac{1}{\pi_{2,6}} + \frac{1}{\pi_{3,7}} + \frac{1}{\pi_{4,8}} = \pi_{\theta,\theta} = 1, \qquad \cdots \quad (151)$$

and, x denoting any small circle,

$$\frac{\pi_{x,5}}{\pi_{1,5}} + \frac{\pi_{x,6}}{\pi_{2,6}} + \frac{\pi_{x,7}}{\pi_{3,7}} + \frac{\pi_{x,8}}{\pi_{4,8}} = 1.$$

139. The system of circles (1, 2, 3, 4) may be called a " semi-orthogonal" system, if (1, 2) cut orthogonally in the points (3, 4). Then x, y denoting any circles we have by

$$\Pi \begin{pmatrix} x, 1, 2, 3, 4 \\ y, 1, 2, 3, 4 \end{pmatrix} = 0,$$

the equation

$$\pi_{x,y} = \frac{\pi_{x,1} \cdot \pi_{y,1}}{\pi_{1,1}} + \frac{\pi_{x,2} \cdot \pi_{y,2}}{\pi_{2,2}} + \frac{\pi_{x,3} \cdot \pi_{y,4} + \pi_{x,4} \cdot \pi_{y,3}}{\pi_{3,4}} . \quad . \quad . \quad . \quad . \quad (152)$$

If $2e$ denote the arc between (3, 4) we have as a particular case,

$$-1 = \cot^2 r_1 + \cot^2 r_2 - \operatorname{cosec}^2 e. \quad . \quad . \quad . \quad . \quad . \quad . \quad (153)$$

Circles touching one another.—§§ 140–144.

140. If the four circles (1, 2, 3, 4) touch one another externally, we shall have from the equation

$$\Pi \begin{pmatrix} \theta, 1, 2, 3, 4 \\ \theta, 1, 2, 3, 4 \end{pmatrix} = 0,$$

$$4 + \cot^2 r_1 + \cot^2 r_2 + \cot^2 r_3 + \cot^2 r_4$$
$$= 2 \{ \cot r_1 \cot r_2 + \cot r_1 \cot r_3 + \cot r_1 \cot r_4$$
$$+ \cot r_2 \cot r_3 + \cot r_2 \cot r_4 + \cot r_3 \cot r_4 \} ;$$

whence

$$\cot r_4 = \cot r_1 + \cot r_2 + \cot r_3 \pm 2 \{ \cot r_2 \cot r_3 + \cot r_3 \cot r_1 + \cot r_1 \cot r_2 - 1 \}^{\frac{1}{2}} . \quad (154)$$

141. We may also easily extend the formulæ (39) and (40) in § 31. Thus, let two circles (1, 2) be described, with angular radii α, γ, and let another circle radius r be described touching these internally, and having its pole on their common diameter. Let S_1 be a circle touching this circle internally, and (1, 2) externally; and let a series of circles S_2, S_3, S_4, &c. be described touching externally (1, 2) and the preceding one in the series; and let the radii of these circles be r_1, r_2, r_3, &c.

We shall have, since S_{n-1} and S_{n+1} touch S_n,

$$\cot r_{n+1} - 2 \cot r_n + \cot r_{n-1} = 2(\cot \alpha + \cot \gamma) ;$$

whence, exactly as in § 31,

$$\cot r_n = n^2(\cot \alpha + \cot \gamma) - \cot r$$

$$= \frac{n^2 \sin r}{\sin \alpha \sin \gamma} - \cot r.$$

So that

$$\tan r_n = \frac{\sin \alpha \sin \gamma \sin r}{n^2 \sin^2 r - \cos r \sin \alpha \sin \gamma} \quad . \quad . \quad . \quad . \quad . \quad (155)$$

Similarly if S'_1, S'_2, S'_3, &c. be a series of circles touching (1, 2) and one another externally, and S'_1 touching the common diameter of (1, 2), we shall find

$$\tan r'_n = \frac{4 \sin \alpha \sin \gamma \sin r}{(2n-1)^2 \sin^2 r - 4 \cos r \sin \alpha \sin \gamma}. \quad . \quad . \quad . \quad . \quad (156)$$

142. Since

$$\pi_{x,y} + \sqrt{\pi_{x,x}.\pi_{y,y}} = \tan r_x \tan r_y (\cos \omega_{x,y} + 1),$$

we infer from § 32 that if the four circles (1, 2, 3, 4) are all touched by another circle externally, then we must have

$$\cos \tfrac{1}{2}\omega_{1,2}.\cos \tfrac{1}{2}\omega_{3,4} \pm \cos \tfrac{1}{2}\omega_{1,3}.\cos \tfrac{1}{2}\omega_{2,4} \pm \cos \tfrac{1}{2}\omega_{1,4}.\cos \tfrac{1}{2}\omega_{2,3} = 0,$$

$\omega_{1,2}$ being the angle of intersection of the circles (1, 2).

This formula must be slightly modified if the tangent circle does not touch them all externally : if, for instance, the circles (1, 2) have contact with the tangent circle of opposite nature, then $\cos \tfrac{1}{2}\omega_{1,2}$ must be replaced by $\sin \tfrac{1}{2}\omega_{1,2}$.

143. If this condition be satisfied the radius of the circle touching the circles (1, 2, 3, 4) may be easily found by means of § 138. Thus, let the orthogonal system of (1, 2, 3, 4) be (5, 6, 7, 8) ; and let the contact be external in each case.

Then since

$$\frac{\pi_{x,1}}{\pi_{1,5}} + \frac{\pi_{x,2}}{\pi_{2,6}} + \frac{\pi_{x,3}}{\pi_{3,7}} + \frac{\pi_{x,4}}{\pi_{4,8}} = 1,$$

we shall have

$$\cot r_x = \frac{\tan r_1}{\pi_{1,5}} + \frac{\tan r_2}{\pi_{2,6}} + \frac{\tan r_3}{\pi_{3,7}} + \frac{\tan r_4}{\pi_{4,8}}. \quad . \quad . \quad . \quad (157)$$

144. If the system of circles (1, 2, 3, 4) be such that four other circles (5, 6, 7, 8) can be drawn to touch them all, symmetrically, say let each of the latter touch one of the former internally and the others externally ; e.g., let (5) touch (2, 3, 4) externally : then since

$$\Pi \begin{pmatrix} x, & 1, & 2, & 3, & 4 \\ y, & 5, & 6, & 7, & 8 \end{pmatrix} = 0,$$

where (x, y) denote any other circles, we have

$$-4\pi_{x,y} = 2\Sigma \pi_{y,1}.\pi_{x,5} \cot r_1 \cot r_5 - (\Sigma \pi_{y,1}.\cot r_1)(\Sigma \pi_{x,5}.\cot r_5) ;$$

whence as a particular case,

$$(\cot r_1 + \cot r_2 + \cot r_3 + \cot r_4)(\cot r_5 + \cot r_6 + \cot r_7 + \cot r_8)$$
$$= 4 + 2(\cot r_1 \cot r_5 + \cot r_2 \cot r_6 + \cot r_3 \cot r_7 + \cot r_4 \cot r_8). \quad . \quad . \quad (158)$$

An example of this would be, when (1, 2, 3, 4) are the inscribed and escribed circles of a spherical triangle, and (5, 6, 7, 8) are the corresponding nine-points circles.

CHAPTER III.—CIRCLES CONNECTED WITH A SPHERICAL TRIANGLE.

Regarding a spherical triangle as formed by the arcs of three small circles, most of the theorems concerning the three species of circles, connected with a triangle formed by great circle-arcs, may be readily extended. It will be seen that there is a much greater resemblance than there is between the corresponding formulæ for plane triangles formed by arcs of circles and straight lines. We shall suppose that the circles intersect in the points P, Q, R, P', Q', R', the former points lying within the triangle formed by arcs joining the poles of the circles; and we will call the angles of the triangle P, Q, R—α, β, γ; then the formulæ for any other of the eight triangles which make up the whole figure may be at once written down by changing two of the angles into their supplements. We shall use r_1, r_2, r_3 to denote angular radii of the circles, and r to denote the angular radius of their orthogonal circle.

The Circum-circle of a Triangle.—§§ 145, 146.

145. If x denote the circle which passes through the points P, Q, R, the points of intersection of the circles (1, 2, 3); and if (4) denote the orthogonal circle of the system (1, 2, 3), we shall have, since

$$\Pi\begin{pmatrix} S, 1, 2, 3, \theta \\ S, 1, 2, 3, 4 \end{pmatrix} = 0,$$

exactly as in § 39,

$$\frac{\pi_{4,4} - \pi_{4,x}}{\pi_{4,x}} \cdot \left\{ \frac{\Pi\begin{pmatrix} 1, 2, 3 \\ 1, 2, 3 \end{pmatrix}}{-\pi_{4,4}} \right\}^{\frac{1}{2}} = \left\{ \Pi\begin{pmatrix} 2, 3 \\ 2, 3 \end{pmatrix} \right\}^{\frac{1}{2}} + \left\{ \Pi\begin{pmatrix} 3, 1 \\ 3, 1 \end{pmatrix} \right\}^{\frac{1}{2}} + \left\{ \Pi\begin{pmatrix} 1, 2 \\ 1, 2 \end{pmatrix} \right\}^{\frac{1}{2}}.$$

But

$$\Pi\begin{pmatrix} 1, 2, 3 \\ 1, 2, 3 \end{pmatrix} = -\frac{36}{R^6} \pi_{4,4} \{ V(1, 2, 3) \}^2 \sec^2 r_1 \sec^2 r_2 \sec^2 r_3,$$

$$\Pi\begin{pmatrix} 2, 3 \\ 2, 3 \end{pmatrix} = \frac{36}{R^6} \sec^2 r_2 \sec^2 r_3 \cdot \{ V(P, 2, 3) \}^2,$$

where R is the radius of the sphere, and V(1, 2, 3) denotes the volume of the tetrahedron formed by the poles of the circles (1, 2, 3) and the centre of the sphere.

Hence we obtain

$$\frac{\pi_{4,4}-\pi_{4,x}}{\pi_{4,x}}=\frac{V(P,\,2,\,3)\cos r_1+V(P,\,3,\,1)\cos r_2+V(P,\,1,\,2)\cos r_3}{V(1,\,2,\,3)};$$

or if ω be the angle of intersection of the circle PQR with the orthogonal circle to (1, 2, 3), we may write

$$\frac{\tan r_x}{\tan r \sec \omega}=\frac{V(1,\,2,\,3)}{V(1,\,2,\,3)+V(P,\,2,\,3)\cos r_1+V(Q,\,3,\,1)\cos r_2+V(R,\,1,\,2)\cos r_3}\quad.\quad(159)$$

146. Again, if $\mu_{1,1}$, $\mu_{1,2}$, &c., denote the minors of $\pi_{1,1}$, $\pi_{1,2}$ &c., in $\Pi\begin{pmatrix}1,\,2,\,3\\1,\,2,\,3\end{pmatrix}$, we shall have, as in § 39,

$$\frac{\pi_{x,x}\,\pi_{4,4}-\pi^2_{4,x}}{\pi^2_{4,x}}\cdot\left\{\Pi\begin{pmatrix}1,\,2,\,3\\1,\,2,\,3\end{pmatrix}\right\}^2=\begin{vmatrix}0,&\sqrt{\mu_{1,1}},&\sqrt{\mu_{2,2}},&\sqrt{\mu_{3,3}}\\\sqrt{\mu_{1,1}},&\mu_{1,1},&\mu_{1,2},&\mu_{1,3}\\\sqrt{\mu_{2,2}},&\mu_{2,1},&\mu_{2,2},&\mu_{2,3}\\\sqrt{\mu_{3,3}},&\mu_{3,1},&\mu_{3,2},&\mu_{3,3}\end{vmatrix}\quad.\quad(160)$$

But

$$\mu_{1,1}=\Pi\begin{pmatrix}2,\,3\\2,\,3\end{pmatrix}=\frac{36}{R^6}\sec^2 r_2\sec^2 r_3\{V(P,\,2,\,3)\}^2;$$

hence

$$\frac{36}{R^6}\cdot\tan^2\omega\sec^2 r_1\sec^2 r_2\sec^2 r_3\tan^4 r^2.\{V(1,\,2,\,3)\}^4$$

$$=\begin{vmatrix}0,&V(P,\,2,\,3)\cos r_1,&V(Q,\,3,\,1)\cos r_2,&V(R,\,1,\,2)\cos r_3\\V(P,\,2,\,3)\cos r_1,&\mu_{1,1},&\mu_{1,2},&\mu_{1,3}\\V(Q,\,3,\,1)\cos r_2,&\mu_{2,1},&\mu_{2,2},&\mu_{2,3}\\V(R,\,1,\,2)\cos r_3,&\mu_{3,1},&\mu_{2,3},&\mu_{3,3}\end{vmatrix};$$

whence may be deduced,

$$\cos^2\omega=\sec s.\cos(s-\alpha).\cos(s-\beta).\cos(s-\gamma),\quad.\quad.\quad.\quad.\quad(161)$$

where

$$2s=\alpha+\beta+\gamma.$$

If the three given circles are great circles, then the imaginary circle θ will be their orthogonal circle, in this case equation (160) reduces to,

$$-\cot^2 r_x=\sec s.\cos(s-\alpha).\cos(s-\beta).\cos(s-\gamma);$$

the ordinary formula for finding the radius of the circum-circle of a spherical triangle

The Inscribed and Escribed Circles of a Triangle.—§§ 147, 148.

147. If the inscribed and escribed circles of the triangle PQR cut the orthogonal circle at angles ω, ω_1, ω_2, ω_3, we have, as in § 43,

$$\left.\begin{aligned}
K.\cos^2 \omega &= 2(1+\cos \alpha)(1+\cos \beta)(1+\cos \gamma) \\
K.\cos^2 \omega_1 &= 2(1+\cos \alpha)(1-\cos \beta)(1-\cos \gamma) \\
K.\cos^2 \omega_2 &= 2(1-\cos \alpha)(1+\cos \beta)(1-\cos \gamma) \\
K.\cos^2 \omega_3 &= 2(1-\cos \alpha)(1-\cos \beta)(1+\cos \gamma)
\end{aligned}\right\} ; \quad . \quad . \quad . \quad (162)$$

where

$$\begin{aligned}
K &= \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma + 2\cos \alpha \cos \beta \cos \gamma - 1 \\
&= 4 \cos s.\cos (s-\alpha).\cos (s-\beta).\cos (s-\gamma).
\end{aligned}$$

In these formulæ $\cos^2 \omega$ has been written for $\dfrac{\pi_{4,x}^2}{\pi_{4,4}.\pi_{x,x}}$, and so, if the given triangle be an ordinary spherical triangle, $\cos^2 \omega$ must be replaced by $-\cot^2 r_x$; thus the above formulæ correspond, in the case of an ordinary spherical triangle, to the formulæ,

$$\left.\begin{aligned}
N \cot r &= 2 \cos \tfrac{1}{2}\alpha \cos \tfrac{1}{2}\beta \cos \tfrac{1}{2}\gamma \\
N \cot r_1 &= 2 \cos \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\beta \sin \tfrac{1}{2}\gamma \\
N \cot r_2 &= 2 \sin \tfrac{1}{2}\alpha \cos \tfrac{1}{2}\beta \sin \tfrac{1}{2}\gamma \\
N \cot r_3 &= 2 \sin \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\beta \cos \tfrac{1}{2}\gamma
\end{aligned}\right\} ;$$

where

$$N^2 = -\cos s.\cos (s-\alpha).\cos (s-\beta).\cos (s-\gamma).$$

In our present case, the radii will be given by formulæ similar to

$$K(\cot r_x + \cos \omega \cot r) + \begin{vmatrix} 0, & \cot r_1, & \cot r_2, & \cot r_3 \\ 1, & -1, & \cos \gamma, & \cos \beta \\ 1, & \cos \gamma, & -1, & \cos \alpha \\ 1, & \cos \beta, & \cos \alpha, & -1 \end{vmatrix} = 0. \quad . \quad . \quad . \quad (163)$$

148. In exactly the same way as in § 45 we may show that, associated with every triangle, there are eight circles analogous to the nine-points circle of a plane triangle, each of them touching four of the circles, which touch the sides of the spherical triangle; that is, taking any one of the eight associated triangles formed by three circles, say P Q R, the inscribed and escribed circles are touched by another circle. If this circle cut the orthogonal circle of the triangle at the angle ϖ, we shall have, as in § 46,

$$\cos^2 \varpi = 4 \sec s.\cos (s-\alpha).\cos (s-\beta).\cos (s-\gamma), \quad . \quad . \quad . \quad (164)$$

and we also infer, from § 46, that this circle cuts the sides of the triangle PQR at the angles $\beta-\gamma$, $\gamma-\alpha$, $\alpha-\beta$.

Comparing equation (164) with equation (161) we see that if the circle PQR cut the orthogonal circle at the angle ω, then $\cos \varpi = 2 \cos \omega$. Whence we infer that in a spherical triangle formed by great circle arcs, the radius ρ of the nine-points circle, and the radius R of the circum-circle, are connected by the formula

$$\cot \rho = 2 \cot R.$$

In the case of a general spherical triangle, this is replaced by the formula

$$\cot \rho - \cot \rho' = 2(\cot R - \cot R'),$$

where ρ, R are the radii of the analogous circles connected with the triangle, and ρ', R' the radii of the corresponding circles connected with the inverse triangle, with respect to the orthogonal circle of the triangle.

CHAPTER IV.—POWER-COORDINATES.

Definition.—§§ 149–151.

149. Given any system of circles, say (1, 2, 3, 4), on the surface of a sphere, then any circle (great or small), or any point, is completely determinate when its powers with respect to the system (1, 2, 3, 4) are known, provided that this system be not a system having a common orthogonal circle.

If then P be any point, we may define the coordinates of P referred to the system (1, 2, 3, 4) as any multiples, the same or different, of the powers of P with respect to these circles; thus denoting the coordinates of P by $(xyzw)$, then k_1, k_2, k_3, k_4 being any constant multiples, we may take

$$x = k_1 . \pi_{P,1}, \quad y = k_2 . \pi_{P,2}, \quad z = k_3 . \pi_{P,3}, \quad w = k_4 . \pi_{P,4}.$$

Since $\pi_{P,P} = 0$, and $\pi_{P,\theta} = 1$, we see at once that the coordinates of any point must satisfy a homogeneous quadric relation, viz.,

$$\Pi \binom{P, 1, 2, 3, 4}{P, 1, 2, 3, 4} = 0,$$

and a non-homogeneous linear relation,

$$\Pi \binom{P, 1, 2, 3, 4}{\theta, 1, 2, 3, 4} = 0.$$

The former is called the equation of the Absolute, and will be usually denoted by $\psi(x, y, z, w)$, and then the latter may be written

or
$$k_1 \frac{\partial \psi}{\partial x} + k_2 \frac{\partial \psi}{\partial y} + k_3 \frac{\partial \psi}{\partial z} + k_4 \frac{\partial \psi}{\partial w} = K$$
$$x \frac{\partial \psi}{\partial k_1} + y \frac{\partial \psi}{\partial k_2} + z \frac{\partial \psi}{\partial k_3} + w \frac{\partial \psi}{\partial k_4} = K$$
$$\Bigg\} \; ; \quad \ldots \ldots \quad (164^*)$$

where $\dfrac{\partial \psi}{\partial k_1}$, as usual, means the partial differential coefficient of ψ with respect to x,

(k_1, k_2, k_3, k_4) being afterwards put for (x, y, z, w), and where K is some constant.

150. If S be any small circle, we may define the coordinates of S with respect to the system $(1, 2, 3, 4)$ as constant multiples of the powers; thus, denoting them by ξ, η, ζ, ω, we will take

$$\xi = k_1 \pi_{S,1}, \quad \eta = k_2 . \pi_{S,2}, \quad \zeta = k_3 . \pi_{S,3}, \quad \omega = k_4 . \pi_{S,4};$$

k_1, k_2, k_3, k_4 having the same values as in § 149.

Since $\pi_{S,\theta} = 1$, we see that the coordinates of any small circle must satisfy the non-homogeneous linear relation

$$k_1 \frac{\partial \psi}{\partial \xi} + k_2 \frac{\partial \psi}{\partial \eta} + k_3 \frac{\partial \psi}{\partial \zeta} + k_4 \frac{\partial \psi}{\partial \omega} = K. \quad \ldots \ldots \quad (165)$$

151. If, however, S be a great circle, we shall have, since $\pi_{S,\theta} = 0$, $\pi_{S,S} = -1$, the homogeneous linear relation,

$$k_1 \frac{\partial \psi}{\partial \xi} + k_2 \frac{\partial \psi}{\partial \eta} + k_3 \frac{\partial \psi}{\partial \zeta} + k_4 \frac{\partial \psi}{\partial \omega} = 0, \quad \ldots \ldots \quad (166)$$

and the non-homogeneous quadric relation,

$$2\psi(\xi, \eta, \zeta, \omega) = -K. \quad \ldots \ldots \quad (167)$$

The Small Circle.—§§ 152–157.

152. If P be any point on the circle S whose coordinates are $(\xi, \eta, \zeta, \omega)$, we shall have by the equation

$$\Pi \binom{P, 1, 2, 3, 4}{S, 1, 2, 3, 4} = 0,$$

since $\pi_{P,S} = 0$,

$$\frac{\partial \psi}{\partial \xi} x + \frac{\partial \psi}{\partial \eta} y + \frac{\partial \psi}{\partial \zeta} z + \frac{\partial \psi}{\partial \omega} w = 0. \quad \ldots \ldots \quad (168)$$

Thus the equation of a small circle is of the first degree.

It follows that the equation of the first degree, say

$$ax + by + cz + dw = 0,$$

will represent in general a small spherical circle, whose coordinates are given by

$$\frac{\dfrac{\partial \psi}{\partial \xi}}{a} = \frac{\dfrac{\partial \psi}{\partial \eta}}{b} = \frac{\dfrac{\partial \psi}{\partial \zeta}}{c} = \frac{\dfrac{\partial \psi}{\partial \omega}}{d} = \frac{K}{ak_1 + bk_2 + ck_3 + dk_4}, \quad . \quad . \quad . \quad . \quad (169)$$

by equation (165).

153. Given any two circles whose coordinates are $(\xi, \eta, \zeta, \omega)$, $(\xi', \eta', \zeta', \omega')$, their power π is given by

$$\Pi \binom{S, \ 1, \ 2, \ 3, \ 4}{S', \ 1, \ 2, \ 3, \ 4} = 0,$$

or

$$\pi . K = \xi' \frac{\partial \psi}{\partial \xi} + \eta' \frac{\partial \psi}{\partial \eta} + \zeta' \frac{\partial \psi}{\partial \zeta} + \omega' \frac{\partial \psi}{\partial \omega} ; \quad . \quad . \quad . \quad . \quad (170)$$

and the radius r of the circle $(\xi \eta \zeta \omega)$ will consequently be given by,

$$- \tan^2 r . K = 2 \psi(\xi, \eta, \zeta, \omega) . \quad . \quad . \quad . \quad . \quad . \quad (171)$$

154. It follows that the radius of the circle

$$ax + by + cz + dw = 0,$$

will be given by

$$\begin{vmatrix} a_{1,1}, & a_{1,2}, & a_{1,3}, & a_{1,4}, & a \\ a_{2,1}, & a_{2,2}, & a_{2,3}, & a_{2,4}, & b \\ a_{3,1}, & a_{3,2}, & a_{3,3}, & a_{3,4}, & c \\ a_{4,1}, & a_{4,2}, & a_{4,3}, & a_{4,4}, & d \\ a, & b, & c, & d, & M \tan^2 r \end{vmatrix} = 0 ; \quad . \quad . \quad . \quad (172)$$

where

$$M = -\frac{2}{K}(ak_1 + bk_2 + ck_3 + dk_4)^2,$$

and where $a_{1,1}$, $a_{1,2}$, &c., are the coefficients in the equation of the absolute, so that

$$\psi(xyzw) \equiv a_{1,1}x^2 + 2a_{1,2}xy + \ldots$$

155. Again, the power of the circle $(\xi \eta \zeta \omega)$ with respect to the circle

$$ax + by + cz + dw = 0,$$

is clearly given by

$$\pi = \frac{a\xi + b\eta + c\zeta + d\omega}{ak_1 + bk_2 + ck_3 + dk_4} . \quad . \quad . \quad . \quad . \quad . \quad . \quad (173)$$

156. And further, the power of the two circles,

$$ax + by + cz + dw = 0,$$
$$a'x + b'y + c'z + d'w = 0,$$

will be given by

$$
\begin{vmatrix}
a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a \\
a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & b \\
a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & c \\
a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & d \\
a' & b' & c' & d' & M\pi
\end{vmatrix} = 0 ; \quad \ldots \quad (174)
$$

where

$$M = \frac{2}{K}(ak_1 + bk_2 + ck_3 + dk_4)(a'k_1 + b'k_2 + c'k_3 + d'k_4).$$

Whence, if the two circles cut at an angle ϕ, we have

$$\cos\phi = -\frac{a'\dfrac{\partial\Psi}{\partial a} + b'\dfrac{\partial\Psi}{\partial b} + c'\dfrac{\partial\Psi}{\partial c} + d'\dfrac{\partial\Psi}{\partial d}}{2\sqrt{\Psi(a, b, c, d).\Psi(a, b, c, d)}} ; \quad \ldots \ldots (175)$$

where

$$
\Psi(a, b, c, d) = -
\begin{vmatrix}
a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a \\
a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & b \\
a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & c \\
a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & d \\
a & b & c & d & 0
\end{vmatrix} .
$$

157. The coordinates of θ, the imaginary circle at infinity, are evidently k_1, k_2, k_3, k_4; and the equation of this circle is

$$x\frac{\partial\psi}{\partial k_1} + y\frac{\partial\psi}{\partial k_2} + z\frac{\partial\psi}{\partial k_3} + w\frac{\partial\psi}{\partial k_4} = 0 ; \quad \ldots \ldots (176)$$

its radius is equal to $\tan^{-1}\sqrt{-1}$.

The Great Circle.—§§ 158, 159.

158. The equation of the first degree

$$ax + by + cz + dw = 0,$$

will represent a great circle on the sphere, when

$$ak_1 + bk_2 + ck_3 + dk_4 = 0 ; \quad \ldots \ldots (177)$$

and if this condition be satisfied, its coordinates will be given by

$$\frac{\dfrac{\partial\psi}{\partial\xi}}{a}=\frac{\dfrac{\partial\psi}{\partial\eta}}{b}=\frac{\dfrac{\partial\psi}{\partial\zeta}}{c}=\frac{\dfrac{\partial\psi}{\partial\omega}}{d}=\sqrt{\frac{2\mathrm{K}\Delta}{-\Psi}}, \quad \ldots \ldots \quad (178)$$

by equation (167), where Ψ has the same meaning as in § 156, and where

$$\Delta = \begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{vmatrix}.$$

159. The power of the great circle $(\xi\eta\zeta\omega)$ with respect to the small circle

$$ax+by+cz+dw=0,$$

is given by

$$\pi\mathrm{K}=\frac{a\xi+b\eta+c\zeta+d\omega}{ak_1+bk_2+ck_3+dk_4},$$

as in § 155.

But if the equation $ax+by+cz+dw=0$ represent a great circle, then the power of any other circle $(\xi\eta\zeta\omega)$ with respect to it is

$$= \sqrt{\frac{2\mathrm{K}\Delta}{-\Psi}}.(a\xi+b\eta+c\zeta+d\omega). \quad \ldots \ldots \quad (179)$$

The angle between two great circles whose equations are given is the same as that given by equation (175).

The Point.—§§ 160–162.

160. The power of the point $(xyzw)$ with respect to the circle

$$ax+by+cz+dw=0,$$

is equal to

$$\frac{ax+by+cz+dw}{ak_1+bk_2+ck_3+dk_4},$$

or

$$(ax+by+cz+dw)\sqrt{\frac{2\Delta\mathrm{K}}{-\Psi}};$$

according as the equation represents a small circle or a great circle.

161. The power of two points $(xyzw)$, $(x'y'z'w')$ is, by equation (168), given by

$$\pi K = x'\frac{\partial \psi}{\partial x} + y'\frac{\partial \psi}{\partial y} + z'\frac{\partial \psi}{\partial z} + w'\frac{\partial \psi}{\partial w} ;$$

which, since

$$\psi(xyzw) = 0, \quad \psi(x'y'z'w') = 0,$$

may be written

$$-2\pi \bar{K} = \psi\{(x-x'), \quad y-y', \quad z-z', \quad w-w'\} . . \quad . \quad . \quad . \quad . \quad (180)$$

162. If R be the radius of the sphere, and (A, B, C) any three points, we have by equation (141),

$$\Pi\binom{\theta, \text{ A, B, C}}{\theta, \text{ A, B, C}} = -\frac{36}{R^6} \cdot \{V(A, B, C)\}^2 ;$$

and by § 132

$$\Pi\binom{\theta, \text{ A, B, C}}{\theta, \text{ A, B, C}} \times \Pi\binom{1, 2, 3, 4}{1, 2, 3, 4} = \left\{\Pi\binom{\theta, \text{ A, B, C}}{1, 2, 3, 4}\right\}^2 .$$

Hence, if $(x_1 y_1 z_1 w_1)$, $(x_2 y_2 z_2 w_2)$, $(x_3 y_3 z_3 w_3)$ be the coordinates of A, B, C, referred to the system (1, 2, 3, 4), we shall have

$$V(ABC) = \mu \begin{vmatrix} x_1, & x_2, & x_3, & k_1 \\ y_1, & y_2, & y_3, & k_2 \\ z_1, & z_2, & z_3, & k_3 \\ w_1, & w_2, & w_3, & k_4 \end{vmatrix} ; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (181)$$

where

$$36\mu^2 k_1^2 k_2^2 k_3^2 k_4^2 . \Pi\binom{1, 2, 3, 4}{1, 2, 3, 4} + R^6 = 0.$$

Coordinate Systems of Reference.—§§ 163, 164.

163. There are two convenient systems of reference : (i.) four mutually orthotomic circles, called the orthogonal system ; (ii.) two orthogonal circles and their two points of intersection, called the semi-orthogonal system. A particular case of the former would be three great circles cutting orthogonally and the imaginary circle at infinity.

(i.) If the system (1, 2, 3, 4) be an orthogonal system, we shall find it most convenient to take k_1, k_2, k_3, k_4 equal respectively to the cotangents of the radii of the circles (1, 2, 3, 4). So that the equation of the absolute will be

$$\psi(x, y, z, w) = x^2 + y^2 + z^2 + w^2 = 0 ;$$

and referring to § 137, we see that we shall have

$$K = -2,$$

and

$$\Psi(a, b, c, d) = a^2 + b^2 + c^2 + d^2 ;$$

also we have

$$\cot^2 r_1 + \cot^2 r_2 + \cot^2 r_3 + \cot^2 r_4 = -1.$$

(ii.) If the system (1, 2, 3, 4) be a semi-orthogonal system, we may take $k_1 = \cot r_1$, $k_2 = \cot r_2$, $k_3 = k_4 = \operatorname{cosec} e$, where r_1, r_2 are the radii of the circles, $2e$ their common chord. We shall have by § 149,

$$\psi(xyzw) = x^2 + y^2 - zw,$$
$$K = -2,$$
$$\Psi(abcd) = a^2 + b^2 - \tfrac{1}{4}cd\,;$$

and also

$$\cot^2 r_1 + \cot^2 r_2 - \cot^2 e = 0.$$

164. By § 128, if two circles S, S′ be inverted with respect to any point (O) on the sphere, then the expression

$$\frac{\pi_{\mathrm{S,\,S'}}}{\sqrt{\pi_{\mathrm{O,\,S}} \cdot \pi_{\mathrm{O,\,S'}}}}$$

is unaltered. Hence, if $(xyzw)$ be the coordinates of any point referred to a system (1, 2, 3, 4), and $(XYZW)$ be the coordinates of the inverse point, with respect to any point (O), referred to the inverse system with respect to the same point, we must have

$$x = \alpha X, \quad y = \beta Y, \quad z = \gamma Z, \quad w = \gamma W\,;$$

and if ξ, η, ζ, ω be the coordinates of any circle, the coordinates of the corresponding circle referred to the new system will be $\alpha\xi$, $\beta\eta$, $\gamma\zeta$, $\delta\omega$: $\alpha\beta\gamma\delta$ being some constants.

CHAPTER V.— GENERAL EQUATION OF THE SECOND DEGREE IN POWER-COORDINATES.

Nature of the Curve.—§ 165.

165. The most general form of the equation of the second degree may be written

$$\phi(x, y, z, w) = ax^2 + by^2 + cz^2 + dw^2 + 2fyz + 2gzx + 2hxy + 2lxw + 2myw + 2nzw = 0, \quad (182)$$

$(xyzw)$ being the coordinates of a point, and therefore satisfying the equation of the absolute ψ, which is also of the second degree; it follows, then, that the general equation of the second degree contains only eight arbitrary constants.

Let P be any point on the curve, and let the Cartesian coordinates of P referred to rectangular axes through the centre of the sphere be X, Y, Z; and let R be the radius of the sphere, then by § 125 we see that we may put

$$x = X^2 + Y^2 + Z^2 - a_1 X - b_1 Y - c_1 Z,$$
$$y = X^2 + Y^2 + Z^2 - a_2 X - b_2 Y - c_2 Z,$$
$$z = X^2 + Y^2 + Z^2 - a_3 X - b_3 Y - c_3 Z,$$
$$w = X^2 + Y^2 + Z^2 - a_4 X - b_4 Y - c_4 Z.$$

Substituting for (x, y, z, w) in equation (182), we see that the curve in question is the curve of intersection of the surfaces

$$X^2 + Y^2 + Z^2 = R^2,$$

and

$$(X^2 + Y^2 + Z^2)^2 + U_1(X^2 + Y^2 + Z^2) + U_2 = 0,$$

U_1 and U_2 being homogeneous expressions of the first and second degree respectively in (X, Y, Z).

So that the curve is the complete intersection of a sphere with a quadric surface, and therefore may be called a "spheri-quadric." These curves have been extensively studied. CASEY calls them sphero-quartics ("Cyclides and Sphero-quartics," (1871). 'Phil. Trans.,' vol. 161). DARBOUX calls them spherical cyclics ('Sur une Classe remarquable de Courbes et de Surfaces Algébriques,' 1873). Mr. H. M. JEFFERY ('London Math. Soc. Proc.,' vol. 20, 1885, p. 102) has proposed to call them sphero-cyclides. The name spheri-quadric is due to Professor CAYLEY.

Equation to Tangent at any Point.—§§ 166–171.

166. Let $(\xi\eta\zeta\omega)$ be the coordinates of any circle touching the spheri-quadric $\phi(xyzw) = 0$ at the point $(x'y'z'w')$, then $\psi(xyzw) = 0$ being the equation of the absolute, we must have

$$\frac{\partial\psi}{\partial\xi} x' + \frac{\partial\psi}{\partial\eta} y' + \frac{\partial\psi}{\partial\zeta} z' + \frac{\partial\psi}{\partial\omega} w' = 0,$$

$$\frac{\partial\psi}{\partial\xi}\delta x' + \frac{\partial\psi}{\partial\eta}\delta y' + \frac{\partial\psi}{\partial\zeta}\delta z' + \frac{\partial\psi}{\partial\omega}\delta w' = 0,$$

$$\frac{\partial\phi}{\partial x'}\delta x' + \frac{\partial\phi}{\partial y'}\delta y' + \frac{\partial\phi}{\partial z'}\delta z + \frac{\partial\phi}{\partial w'}\delta w' = 0,$$

$$\frac{\partial\psi}{\partial x'}\delta x' + \frac{\partial\psi}{\partial y'}\delta y' + \frac{\partial\psi}{\partial z'}\delta z + \frac{\partial\psi}{\partial w'}\delta w' = 0.$$

And hence we must have

$$\frac{\dfrac{\partial\psi}{\partial\xi}}{\dfrac{\partial\phi}{\partial x'} + k\dfrac{\partial\psi}{\partial x'}} = \frac{\dfrac{\partial\psi}{\partial\eta}}{\dfrac{\partial\phi}{\partial y'} + k\dfrac{\partial\psi}{\partial y'}} = \frac{\dfrac{\partial\psi}{\partial\zeta}}{\dfrac{\partial\phi}{\partial z'} + k\dfrac{\partial\psi}{\partial z'}} = \frac{\dfrac{\partial\psi}{\partial\omega}}{\dfrac{\partial\phi}{\partial w'} + k\dfrac{\partial\psi}{\partial w'}}, \quad \ldots \ldots \quad (183)$$

where k is indeterminate.

Hence every circle which touches the curve at the point $(x'y'z'w')$ has its equation of the form

$$\left(x\frac{\partial}{\partial x'} + y\frac{\partial}{\partial y'} + z\frac{\partial}{\partial z'} + w\frac{\partial}{\partial w'}\right)(\phi + k\psi) = 0. \quad \ldots \ldots \quad (184)$$

167. To determine the equation of the tangent (*i.e.*, tangent great circle) to the curve at the point $(x'y'z'w')$, we must determine k in equation (184), so that the equation may be satisfied by $(k_1 k_2 k_3 k_4)$ the coordinates of the circle at infinity. Thus the equation of the tangent is

$$\left(k_1\frac{\partial\phi}{\partial x'}+k_2\frac{\partial\phi}{\partial y'}+k_3\frac{\partial\phi}{\partial z'}+k_4\frac{\partial\phi}{\partial w'}\right)\left(x\frac{\partial\psi}{\partial x'}+y\frac{\partial\psi}{\partial y'}+z\frac{\partial\psi}{\partial z'}+w\frac{\partial\psi}{\partial w'}\right)$$
$$=\left(k_1\frac{\partial\psi}{\partial x'}+k_2\frac{\partial\psi}{\partial y'}+k_3\frac{\partial\psi}{\partial z'}+k_4\frac{\partial\psi}{\partial w'}\right)\left(x\frac{\partial\phi}{\partial x'}+y\frac{\partial\phi}{\partial y'}+z\frac{\partial\phi}{\partial z'}+w\frac{\partial\phi}{\partial w'}\right). \quad \cdots \quad (185)$$

168. The circle given by equations (183) or (184) will touch the curve at the point $(x''y''z''w'')$ if

$$\frac{\frac{\partial\phi}{\partial x'}+k\frac{\partial\psi}{\partial x'}}{\frac{\partial\phi}{\partial x''}+k\frac{\partial\psi}{\partial x''}}=\frac{\frac{\partial\phi}{\partial y'}+k\frac{\partial\psi}{\partial y'}}{\frac{\partial\phi}{\partial y''}+k\frac{\partial\psi}{\partial y''}}=\frac{\frac{\partial\phi}{\partial z'}+k\frac{\partial\psi}{\partial z'}}{\frac{\partial\phi}{\partial z''}+k\frac{\partial\psi}{\partial z''}}=\frac{\frac{\partial\phi}{\partial w'}+k\frac{\partial\psi}{\partial w'}}{\frac{\partial\phi}{\partial w''}+k\frac{\partial\psi}{\partial w''}};$$

i.e., if k satisfy the quartic equation

$$\begin{vmatrix} \frac{\partial^2}{\partial x^2} & \frac{\partial^2}{\partial x\partial y} & \frac{\partial^2}{\partial x\partial z} & \frac{\partial^2}{\partial x\partial w} \\ \frac{\partial^2}{\partial x\partial y} & \frac{\partial^2}{\partial y^2} & \frac{\partial^2}{\partial y\partial z} & \frac{\partial^2}{\partial y\partial w} \\ \frac{\partial^2}{\partial x\partial z} & \frac{\partial^2}{\partial y\partial z} & \frac{\partial^2}{\partial z^2} & \frac{\partial^2}{\partial z\partial w} \\ \frac{\partial^2}{\partial x\partial w} & \frac{\partial^2}{\partial y\partial w} & \frac{\partial^2}{\partial z\partial w} & \frac{\partial^2}{\partial w^2} \end{vmatrix} (\phi+k\psi)=0,$$

i.e.,

$$\mathrm{H}(\phi+k\psi)=0, \quad \cdots \quad \cdots \quad \cdots \quad (186)$$

where $\mathrm{H}(u)$ denotes the Hessian of u.

We infer then that there are, in general, four systems of bitangent circles, each circle belonging to a particular system cutting a certain circle orthogonally; the coordinates of these four circles being proportional to the minors of the constituents of any row of the above determinants, corresponding to the four values of k.

169. If the coordinates of a bitangent circle satisfy the condition

$$k_1\frac{\partial\psi}{\partial\xi}+k_2\frac{\partial\psi}{\partial\eta}+k_3\frac{\partial\psi}{\partial\zeta}+k_4\frac{\partial\psi}{\partial\omega}=0,$$

the circle is a great circle; there will, in general, be eight such great circles, two belonging to each system of bitangent circles.

170. If the coordinates of a bitangent circle satisfy the equation of the absolute, the circle reduces to a point and corresponds to a focus of a plane bicircular quartic—there are clearly sixteen such foci, four on each of the circles which cut the bitangent systems orthogonally. Dr. CASEY ("Cyclides and Sphero-Quartics") calls these single-foci.

171. It is clear that, if by any linear transformation of coordinates the equations $\phi=0$, $\psi=0$ become respectively $\Phi=0$, $\Psi=0$, then the same value of k which satisfies $H(\phi+k\psi)=0$ must also satisfy $H(\Phi+k\Psi)=0$.

Hence the coefficients of powers of k in the equation (186) are invariants.

Equation of the Normal at any Point.—§§ 172–174.

172. Let $(\xi\eta\zeta\omega)$ be the coordinates of any circle which cuts the curve $\phi(xyzw)=0$ orthogonally at the point $(x'y'z'w')$, then we must have

$$\left(\xi\frac{\partial}{\partial x'}+\eta\frac{\partial}{\partial y'}+\zeta\frac{\partial}{\partial z'}+\omega\frac{\partial}{\partial w'}\right)(\phi+k\psi)=0 \quad . \quad . \quad . \quad . \quad . \quad (187)$$

for all values of k.

173. It follows that the coordinates of the normal (*i.e.*, great circle) at $(x'y'z'w')$ must satisfy

$$\xi\frac{\partial\phi}{\partial x'}+\eta\frac{\partial\phi}{\partial y'}+\zeta\frac{\partial\phi}{\partial z'}+\omega\frac{\partial\phi}{\partial w'}=0,$$

$$\xi\frac{\partial\psi}{\partial x'}+\eta\frac{\partial\psi}{\partial y'}+\zeta\frac{\partial\psi}{\partial z'}+\omega\frac{\partial\psi}{\partial w'}=0,$$

$$\xi\frac{\partial\psi}{\partial k_1}+\eta\frac{\partial\psi}{\partial k_2}+\zeta\frac{\partial\psi}{\partial k_3}+\omega\frac{\partial\psi}{\partial k_4}=0.$$

Hence the equation of the normal is

$$\begin{vmatrix} \dfrac{\partial\psi}{\partial x}, & \dfrac{\partial\psi}{\partial y}, & \dfrac{\partial\psi}{\partial z}, & \dfrac{\partial\psi}{\partial w} \\[2ex] \dfrac{\partial\phi}{\partial x'}, & \dfrac{\partial\phi}{\partial y'}, & \dfrac{\partial\phi}{\partial z'}, & \dfrac{\partial\phi}{\partial w'} \\[2ex] \dfrac{\partial\psi}{\partial x'}, & \dfrac{\partial\psi}{\partial y'}, & \dfrac{\partial\psi}{\partial z'}, & \dfrac{\partial\psi}{\partial w'} \\[2ex] \dfrac{\partial\psi}{\partial k_1}, & \dfrac{\partial\psi}{\partial k_2}, & \dfrac{\partial\psi}{\partial k_3}, & \dfrac{\partial\psi}{\partial k_4} \end{vmatrix} =0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (188)$$

174. From equation (187) we can deduce

$$x'\frac{\partial\phi}{\partial\xi}+y'\frac{\partial\phi}{\partial\eta}+z'\frac{\partial\phi}{\partial\zeta}+w'\frac{\partial\phi}{\partial\omega}=0,$$

$$x'\frac{\partial\psi}{\partial\xi}+y'\frac{\partial\psi}{\partial\eta}+z'\frac{\partial\psi}{\partial\zeta}+w'\frac{\partial\psi}{\partial\omega}=0.$$

If then $(\xi\eta\zeta\omega)$ be so chosen that

$$\frac{\dfrac{\partial\phi}{\partial\xi}}{\dfrac{\partial\psi}{\partial\xi}}=\frac{\dfrac{\partial\phi}{\partial\eta}}{\dfrac{\partial\psi}{\partial\eta}}=\frac{\dfrac{\partial\phi}{\partial\zeta}}{\dfrac{\partial\psi}{\partial\zeta}}=\frac{\dfrac{\partial\phi}{\partial\omega}}{\dfrac{\partial\psi}{\partial\omega}}=-\mu,\text{ say,}$$

then the circle $(\xi\eta\zeta\omega)$ cuts $\phi(xyzw)$ orthogonally in each of the four points in which it meets it; we see at once that μ must satisfy the equation

$$\mathrm{H}(\phi+\mu\psi)=0,$$

which is the same equation as in § 168; the coordinates of the four orthogonal circles corresponding to the four values of μ, being proportional to the minors of the determinant $\mathrm{H}(\phi+\mu\psi)$.

The Principal Circles.—§§ 175–179.

175. The four orthogonal circles found in the last article are usually called the principal circles of the curve. By § 168, we see that a spheri-quadric is the envelope of a series of circles which cut one of the principal circles orthogonally, and it is evident by inversion that the curve must be anallagmatic, i.e., its own inverse with respect to each of its four principal circles; also each point in which a principal circle cuts a spheri-quadric must be a cyclic point on the curve; there are in general sixteen such points. Again, the double great circle tangents are the tangents which can be drawn from the poles of the principal circles.

176. It is easily proved, as in § 81, that any two circles corresponding to different values of k given by $\mathrm{H}(\phi+k\psi)=0$, cut orthogonally; hence, if the four roots of the discriminating quartic be different there are four principal circles which are mutually orthotomic, and the poles of these circles must be such that the arc joining any two is perpendicular to the arc joining the remaining two.

177. If the roots of $\mathrm{H}(\phi+k\psi)=0$ are all different, then we can reduce the equation to the form

$$ax^2+by^2+cz^2+dw^2=0,$$

the system of reference being the four principal circles, and a, b, c, d being the roots of the discriminating quartic.

178. If two roots of the quartic $\mathrm{H}(\phi+k\psi)$ are equal, then taking the two principal circles corresponding to the two other values of k, and any other circles forming with them an orthogonal system, as circles of reference, we can reduce the equation to the form

$$ax^2+by^2+cz^2+dw^2+2nzw=0,$$

and, exactly as in § 83, we see that if one of the two circles (x, y) be imaginary, then the discriminating quartic can have two equal roots only when

$$c=d,\qquad n=0;$$

in which case the equation reduces to

$$ax^2 + by^2 + cz^2 + cw^2 = 0,$$

which represents two imaginary circles.

But if (x, y) be both real, then by taking for system of reference the two principal circles (x, y), and their two points of intersection (z, w) we can show that the equation may be reduced to the form

$$ax^2 + by^2 + cz^2 = 0,$$

the equation of the absolute being

$$x^2 + y^2 = 4zw.$$

A spheri-quadric represented by an equation of this form has a finite node, viz., the point $z = 0$.

179. Now, let us suppose the discriminating quartic to have three equal roots, then, as in § 84, we can show that if we take as system of reference the principal circle (x) corresponding to the unequal root, the node (z), and the circle (y), passing through (z), cutting (x) orthogonally, and passing through the other point (w), in which (x) cuts the curve : the equation may be reduced to the form

$$x^2 = 2fyz.$$

The point z is clearly a cusp, the circle $x = 0$ being the cuspidal edge.

Observation.—If we suppose two of our circles of reference to be great circles, the curves degenerate into sphero-conics. As from § 128, it is clear that inversion is merely equivalent to a linear transformation, nodal and cuspidal spheri-quadrics are the inverse curves of sphero-conics.

CHAPTER VI.—CLASSIFICATION OF SPHERI-QUADRICS.

The method followed in Part I. for the classification of bicircular quartics is not suited for a systematic classification of spheri-quadrics, for which see CASEY, " On Cyclides and Sphero-Quartics," chap. xi. In this memoir it is only proposed to discuss the chief properties of the curves, following the order of chap. vii., Part I.

General Spheri-quadric.—§§ 180–184*.

180. The equation of the curve is of the form

$$ax^2 + by^2 + cz^2 + dw^2 = 0,$$

and the equation of the absolute may be taken as

$$x^2 + y^2 + z^2 + w^2 = 0 \; ;$$

where the coordinates of the circle at infinity are $\cot r_1$, $\cot r_2$, $\cot r_3$, $\cot r_4$; r_1, r_2, r_3, r_4 being the radii of the principal circles.

181. The coordinates of any circle touching the curve at the point $(x'y'z'w')$ must be proportional to

$$(a+k)x', \quad (b+k)y', \quad (c+k)z', \quad (d+k)w'.$$

The equation to the tangent at the point will be,

$$(x'x+y'y+z'z+w'w)(ax' \cot r_1+by' \cot r_2+cz' \cot r_3+dw' \cot r_4)$$
$$=(ax'x+by'y+cz'z+dw'w)(x' \cot r_1+y' \cot r_2+z' \cot r_3+w' \cot r_4). \qquad (189)$$

The equation to the normal at the point $(x'y'z'w')$ will be,

$$\begin{vmatrix} x, & y, & z, & w \\ x', & y', & z', & w' \\ ax', & by', & cz', & dw' \\ \cot r_1, & \cot r_2, & \cot r_3, & \cot r_4 \end{vmatrix} = 0 \quad . \quad . \quad . \quad . \quad (190)$$

182. The systems of bitangent circles will be given by,

$$\left. \begin{array}{ll} \xi=0, & \dfrac{\eta^2}{b-a}+\dfrac{\zeta^2}{c-a}+\dfrac{\omega^2}{d-a}=0 \\[2ex] \eta=0, & \dfrac{\xi^2}{a-b}+\dfrac{\zeta^2}{c-b}+\dfrac{\omega^2}{d-b}=0 \\[2ex] \zeta=0, & \dfrac{\xi^2}{a-c}+\dfrac{\eta^2}{b-c}+\dfrac{\omega^2}{d-c}=0 \\[2ex] \omega=0, & \dfrac{\xi^2}{a-d}+\dfrac{\eta^2}{b-d}+\dfrac{\zeta^2}{c-d}=0 \end{array} \right\} \; ; \quad . \quad . \quad . \quad . \quad . \quad (191)$$

and the coordinates of the single foci will be given by,

$$\left. \begin{array}{ll} x=0, & \dfrac{y^2}{(c-d)(b-a)}=\dfrac{z^2}{(d-b)(c-a)}=\dfrac{w^2}{(b-c)(d-a)} \\[2ex] y=0, & \dfrac{x^2}{(c-d)(a-b)}=\dfrac{z^2}{(d-a)(c-b)}=\dfrac{w^2}{(a-c)(d-b)} \\[2ex] z=0, & \dfrac{x^2}{(b-d)(a-c)}=\dfrac{y^2}{(d-a)(b-c)}=\dfrac{w^2}{(a-b)(d-c)} \\[2ex] w=0, & \dfrac{x^2}{(b-c)(a-d)}=\dfrac{y^2}{(c-a)(b-d)}=\dfrac{z^2}{(a-b)(c-d)} \end{array} \right\} \quad . \quad . \quad . \quad . \quad (192)$$

The curve has also six double foci (see CASEY, "Cyclides," § 130), and thus the twenty-eight points of intersection of the eight common tangents of the curve and the circle at infinity are accounted for.

183. From the form of the equations (192), it follows that all curves given by

$$\frac{x^2}{\alpha^2+\kappa}+\frac{y^2}{\beta^2+\kappa}+\frac{z^2}{\gamma^2+\kappa}+\frac{w^2}{\delta^2+\kappa}=0, \quad . \quad . \quad . \quad . \quad . \quad (193)$$

must be confocal with the curve

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}+\frac{w^2}{\delta^2}=0.$$

Subtracting these equations, we have

$$\frac{x^2}{\alpha^2(\alpha^2+\kappa)}+\frac{y^2}{\beta^2(\beta^2+\kappa)}+\frac{z^2}{\gamma^2(\gamma^2+\kappa)}+\frac{w^2}{\delta^2(\delta^2+\kappa)}=0.$$

Hence the curves cut orthogonally at their common points.

And since equation (193) may be regarded as a quadratic in κ, we infer that through any point on a sphere two spheri-quadrics can be drawn confocal with a given spheri-quadric, and these two cut orthogonally.

184. We may prove exactly, as in § 123, that the coordinates of the osculating circle at any point $(x'y'z'w')$ on the curve

$$ax^2+by^2+cz^2+dw^2=0,$$

will be proportional to

$$(a-b)(a-c)(a-d)x'^3, \quad (b-a)(b-c)(b-d)y'^3, \quad (c-a)(c-b)(c-d)z'^3,$$
$$(d-a)(d-b)(d-c)w'^3 ; \quad . \quad . \quad . \quad . \quad . \quad (194)$$

and if R be the radius of curvature at the point we shall have by § 154,

$$(a^2x'^2+b^2y'^2+c^2z'^2+d^2w'^2)^{\frac{3}{2}}.\cot R$$
$$=(a-b)(a-c)(a-d)\cot r_1 x'^3+(b-a)(b-c)(b-d)\cot r_2 y'^3$$
$$+(c-a)(c-b)(c-d)\cot r_2 z'^3+(d-a)(d-b)(d-c)\cot r_3 w'^3. \quad . \quad (195)$$

184*. If one of the principal circles is a great circle, the corresponding foci may coincide with the centres of one of the other principal circles; in this case the curve has been called by Casey a "sphero-Cartesian." Thus suppose $r_4=\dfrac{\pi}{2}$, and let one of the foci on $w=0$ coincide with the centre of the circle whose radius is r_1, the coordinates of this point are $-\operatorname{cosec} r_1$, $\cot r_2$, $\cot r_3$; and the necessary condition that the curve

$$ax^2+by^2+cz^2+dw^2=0$$

may represent a " Cartesian " is given by

$$\frac{\operatorname{cosec}^2 r_1}{a-d}+\frac{\cot^2 r_2}{b-d}+\frac{\cot^2 r_3}{c-d}=0. \quad . \quad . \quad . \quad . \quad . \quad (196)$$

Spheri-quadrics having a Third Node.—§§ 185–189.

185. The equation of the curve is of the form

$$ax^2+by^2+cz^2=0,$$

the equation of the absolute being

$$x^2+y^2=zw ;$$

where the coordinates of the circle at infinity are $\cot r_1$, $\cot r_2$, $\operatorname{cosec} e$, $\operatorname{cosec} e$; r_1, r_2 being the radii of the two principal circles, and $2e$ the arc between their points of intersection.

186. The coordinates $(\xi\eta\zeta\omega)$ of any circle which touches the curve at the point $(x'y'z'w')$ are given by

$$\frac{2\xi}{(a+2k)x'}=\frac{2\eta}{(b+2k)y'}=\frac{-\omega}{cz'-kw'}=\frac{-\zeta}{-kz'}.$$

The equation of the tangent at the point will be,

$$(2x'x+2y'y-w'z-z'w)(ax'\cot r_1+by'\cot r_2+cz'\operatorname{cosec} e)$$

$$=(ax'x+by'y+cz'z)(2x'\cot r_1+2y'\cot r_2-w'\operatorname{cosec} e-z'\operatorname{cosec} e). \quad . \quad (197)$$

The equation of the normal at $(x'y'z'w')$ will be,

$$\begin{vmatrix} 2x, & 2y, & -w, & -z \\ 2x', & 2y', & -w', & -z' \\ ax', & by', & cz', & 0 \\ 2\cot r_1, & 2\cot r_2, & -\operatorname{cosec} e, & -\operatorname{cosec} e \end{vmatrix}=0 \quad . \quad . \quad . \quad (198)$$

187. The systems of bitangent circles will be given by

$$\left.\begin{aligned} \xi=0, \quad & \frac{\eta^2}{b-a}+\frac{\zeta\omega}{a}-\frac{c\zeta^2}{a^2}=0 \\ \eta=0, \quad & \frac{\xi^2}{a-b}+\frac{\zeta\omega}{b}-\frac{c\zeta^2}{b^2}=0 \end{aligned}\right\} ; \quad . \quad . \quad . \quad . \quad (199)$$

and the coordinates of the foci by

$$\left. \begin{array}{ll} x=0, & y^2=zw=\dfrac{(b-a)c}{ab}z^2 \\[3mm] y=0, & x^2=zw=\dfrac{(a-b)c}{ab}z^2 \end{array} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (200)$$

188. From the form of these equations we see that every curve whose equation is of the form

$$\frac{x^2}{\alpha^2+\kappa}+\frac{y^2}{\beta^2+\kappa}+\frac{z^2}{\gamma^2}=0,$$

is confocal with the curve

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=0 ;$$

and subtracting we have

$$\frac{x^2}{\alpha^2(\alpha^2+\kappa)}+\frac{y^2}{\beta^2(\beta^2+\kappa)}=0.$$

Hence two such curves intersect orthogonally.

We infer that through any point on a sphere two spheri-quadrics can be drawn confocal with a given nodal-spheri-quadric ; and these curves will cut orthogonally.

189. If $r_2=\dfrac{\pi}{2}$ and the coefficients a, b, c in the equation

$$ax^2+by^2+cz^2=0,$$

satisfy the relation

$$(a-b)c=ab, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (201)$$

then one of the foci on the principal circle $y=0$ coincides with the centre of the other principal circle, and the curve becomes a Cartesian, having a third node.

Cuspidal Spheri-quadrics.—§§ 190–192.

190. The equation of the curve is of the form

$$x^2=2ayz,$$

the system of reference being the principal circle, the circle orthogonal to it through the cusp and the other point, in which the principal circle cuts the curve, the cusp and the other point common to the two circles.

If r_1, r_2 be the radii of the circles, $2e$ the arc between their points of intersection, we may take the equation of the absolute as

$$x^2+y^2=zw,$$

the coordinates of the circle at infinity being $\cot r_1$, $\cot r_2$, $\operatorname{cosec} e$, $\operatorname{cosec} e$.

191. The coordinates $(\xi\eta\zeta\omega)$ of any circle which touches the curve at the point $(x'y'z'w')$ must satisfy

$$\frac{2\xi}{(1+2k)x'} = \frac{2\eta}{-az'+2ky'} = \frac{-\omega}{-ay'-kw'} = \frac{-\zeta}{-kz'}.$$

The equation of the tangent at $(x'y'z'w')$ will be

$$(2x'x+2y'y-w'z-wz')(x'\cot r_1-az'\cot r_2-ay'\cosec e)$$
$$=(x'x-az'y-ay'z)(2x'\cot r_1+2y'\cot r_2-(w'+z')\cosec e).\quad . \quad . \quad (202)$$

The equation of the normal at $(x'y'z'w')$ will be,

$$\begin{vmatrix} 2x, & 2y, & w, & z \\ 2x', & 2y', & w', & z' \\ x', & -az', & ay', & 0 \\ 2\cot r_1, & 2\cot r_2, & \cosec e, & \cosec e \end{vmatrix} = 0. \quad . \quad . \quad . \quad (203)$$

192. The system of bitangent circles is given by

$$\xi=0, \quad (\eta-a\zeta)^2=\zeta\omega . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (204)$$

The focus of the curve is given by

$$\frac{x}{0}=\frac{y}{a}=\frac{z}{2}=\frac{2w}{a^2}.. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (205)$$

Equation of a Spheri-quadric Referred to three Circles Orthogonal to one of its Principal Circles.—§§ 193–198.

193. Let $w=0$ be one of the principal circles of a spheri-quadric; then if (x, y, z) are any three circles orthogonal to w, the equation of the spheri-quadric must be of the form

$$w^2+f(x, y, z)=0,$$

and the equation of the absolute will also be of the same form. Hence, by subtraction, we have for the equation of the spheri-quadric

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0,. \quad . \quad . \quad . \quad (206)$$

and this is a form to which the equation of any spheri-quadric can be reduced.

194. We shall find it convenient to suppose the coordinates (xyz) to be equal to the

powers of a point with respect to the three circles of reference; then by § 136, equation (147), the equation of any circle orthogonal to w will be given by

$$\alpha x + \beta y + \gamma z = 0, \quad \cdots \cdots \cdots \quad (207)$$

where α, β, γ are proportional to the triangular coordinates of the pole of the circle referred to the triangle formed by the poles of the circles (x, y, z).

Suppose, now, the circle given by (206) to be a bitangent circle of the spheri-quadric (205), then we must have

$$\begin{vmatrix} a, & h, & g, & \alpha \\ h, & b, & f, & \beta \\ g, & f, & c, & \gamma \\ \alpha, & \beta, & \gamma, & 0 \end{vmatrix} = 0. \quad \cdots \cdots \cdots \quad (208)$$

Hence it follows that the poles of all bitangent circles belonging to the same system lie on a sphero-conic; or again, the spheri-quadric (205) is the envelope of circles whose poles lie on (208), and which cut a given circle $w=0$ orthogonally.

195. If the circles (x, y, z) are the other three principal circles of the curve (205), we know that the equation of the curve is of the form

$$ax^2 + by^2 + cz^2 = 0,$$

hence the equation of the sphero-conic is

$$\frac{\alpha^2}{a} + \frac{\beta^2}{b} + \frac{\gamma^2}{c} = 0.$$

Thus the sphero-conic corresponding to one principal circle is self-conjugate to the triangle formed by the poles of the other three principal circles.

196. Again, in the case of a nodal-spheri-quadric the equation of the curve referred to its other principal circle, and the two points in which its two principal circles intersect, is of the form

$$ax^2 + by^2 + 2fyz = 0,$$

so that the equation of the sphero-conic must be

$$f^2 \alpha^2 - ab\gamma^2 + 2af\beta\gamma = 0.$$

Thus the sphero-conic must pass through the node.

197. Let (y, z) be any two bitangent circles, (x) the circle which passes through their four points of contact; the equation of the curve takes the form

$$x^2 = 2fyz,$$

and the sphero-conic must be given by

$$f\alpha^2 = 2\beta\gamma.$$

As a particular case, we may suppose (y, z) to be the pair of great circles which can be drawn from the pole of (w) to have double contact with the curve, and then it follows that the centre of the sphero-conic coincides with the centre of the circle which passes through the points of contact of these double tangents : *i.e.*, the centre of the sphero-conic coincides with the centre of the polar circle of the centre of w with respect to the spheri-quadric.

198. Taking for circles of reference any three bitangent circles orthogonal to w, the equation of the curve takes the form

$$a\sqrt{x} + b\sqrt{y} + c\sqrt{z} = 0,$$

and the equation of the sphero-conic becomes

$$\frac{a^2}{\alpha} + \frac{b^2}{\beta} + \frac{c^2}{\gamma} = 0.$$

Hence, if P be any point on the curve, A, B, C three foci on the same principal circle, then

$$a.\sin \tfrac{1}{2}\mathrm{AP} + b.\sin \tfrac{1}{2}\mathrm{BP} + c.\sin \tfrac{1}{2}\mathrm{CP} = 0. \quad . \quad . \quad . \quad . \quad (209)$$

Or, again, if the curve is a sphero-Cartesian, so that the sphero-conic becomes a circle, then A, B, C being the centres of any three bitangent circles of the system, P a point on the curve ; $2p$, $2q$, $2r$ the tangents from P to these circles ; we have

$$\sin \tfrac{1}{2}a.\sin \tfrac{1}{2}p + \sin \tfrac{1}{2}b.\sin \tfrac{1}{2}q + \sin \tfrac{1}{2}c.\sin \tfrac{1}{2}r = 0, \quad . \quad . \quad . \quad (210)$$

a, b, c being the sides of the triangle ABC.

PART III.—SYSTEMS OF SPHERES.

CHAPTER I.—GENERAL SYSTEMS OF SPHERES.

The Power of two Spheres.—§§ 199–201.

199. The power of two spheres is the square of the distance between their centres less the sum of the squares of their radii.

Thus if any two spheres be denoted by (1, 2) we shall have

$$\pi_{1,2} = d^2_{1,2} - r_1^2 - r_2^2 = 2r_1 r_2 \cos \omega_{1,2} :$$

where $\pi_{1,2}$ denotes their power, r_1, r_2 their radii, $\omega_{1,2}$ their angle of intersection, $d_{1,2}$ the distance between their centres.

The definition is due to DARBOUX ('Annales de l'École Normale Supérieure,' vol. 1, 1872); it is also given in a paper to be found in CLIFFORD's 'Mathematical Papers,' p. 332; the date of which paper is assigned by the editor as 1868 (see note on p. 332).

200. If the equations of two spheres be

$$x^2+y^2+z^2+2fx+2gy+2hz+c=0,$$
$$x^2+y^2+z^2+2f'x+2g'y+2h'z+c'=0;$$

we have at once for their power

$$\pi=c+c'-2ff'-2gg'-2hh'. \quad . \quad . \quad . \quad . \quad . \quad . \quad (211)$$

Extending the definition given in § 4, the power of a sphere and a plane may be defined as twice the perpendicular distance of the centre from the plane; thus the power of the sphere

$$x^2+y^2+z^2+2fx+2gy+2hz+c=0,$$

and the plane

$$x\cos\alpha+y\cos\beta+z\cos\gamma-p=0,$$

will be

$$\pi=2p-2f\cos\alpha-2g\cos\beta-2h\cos\gamma. \quad . \quad . \quad . \quad . \quad . \quad (212)$$

And similarly the power of two planes may be defined as twice the cosine of the angle between them.

Also if θ denote the plane at infinity, S any sphere, or point (considered as a sphere of indefinitely small radius), we shall have

$$\pi_{\theta,\,S}=1;$$

and if L be any plane, $\pi_{\theta,\,L}=0$; and also $\pi_{\theta,\,\theta}=0$.

201. If we take the inverse spheres, with respect to a sphere whose centre is the origin and radius R, of the spheres

$$x^2+y^2+z^2+2fx+2gy+2hz+c=0,$$
$$x^2+y^2+z^2+2f'x+2g'y+2h'z+c'=0;$$

we see at once, that the power π' of the inverse spheres is connected with the power of the original spheres by the formula

$$\pi'=\pi\cdot\frac{R^4}{cc'};$$

whence we deduce at once, denoting the spheres by S_1, S_2, and the inverse spheres by S'_1, S'_2,

$$\frac{\pi_{S_1, S_2}}{\sqrt{\pi_{0, S_1} \cdot \pi_{0, S_2}}} = \frac{\pi_{S'_1, S'_2}}{\sqrt{\pi_{0, S'_1} \cdot \pi_{0, S'_2}}},$$

(O) denoting the origin.

And generally, if x, y denote either spheres, points, or planes, we infer that the expression

$$\frac{\pi_{x, y}}{\sqrt{\pi_{0, x} \cdot \pi_{0, y}}}$$

is unaltered by inverting on the point (O).

General Theorems.—§§ 202–205.

202. If we have a system of six spheres, say (1, 2, 3, 4, 5, 6), their powers with respect to any other system of six spheres, say (7, 8, 9, 10, 11, 12), are connected by the relation

$$\Pi \begin{pmatrix} 1, 2, 3, 4, 5, 6 \\ 7, 8, 9, 10, 11, 12 \end{pmatrix} = 0.$$

For if we multiply together the matrices

$$\begin{vmatrix} 1, & 2f_1, & 2g_1, & 2h_1, & c_1 \\ 1, & 2f_2, & 2g_2, & 2h_2, & c_2 \\ 1, & 2f_3, & 2g_3, & 2h_3, & c_3 \\ 1, & 2f_4, & 2g_4, & 2h_4, & c_4 \\ 1, & 2f_5, & 2g_5, & 2h_5, & c_5 \\ 1, & 2f_6, & 2g_6, & 2h_6, & c_6 \end{vmatrix} , \quad \begin{vmatrix} c_7, & -f_7, & -g_7, & -h_7, & 1 \\ c_8, & -f_8, & -g_8, & -h_8, & 1 \\ c_9, & -f_9, & -g_9, & -h_9, & 1 \\ c_{10}, & -f_{10}, & -g_{10}, & -h_{10}, & 1 \\ c_{11}, & -f_{11}, & -g_{11}, & -h_{11}, & 1 \\ c_{12}, & -f_{12}, & -g_{12}, & -h_{12}, & 1 \end{vmatrix} ;$$

we have at once the equation,

$$\begin{vmatrix} \pi_{1,7}, & \pi_{1,8}, & \pi_{1,9}, & \pi_{1,10}, & \pi_{1,11}, & \pi_{1,12} \\ \pi_{2,7}, & \pi_{2,8}, & \pi_{2,9}, & \pi_{2,10}, & \pi_{2,11}, & \pi_{2,12} \\ \pi_{3,7}, & \pi_{3,8}, & \pi_{3,9}, & \pi_{3,10}, & \pi_{3,11}, & \pi_{3,12} \\ \pi_{4,7}, & \pi_{4,8}, & \pi_{4,9}, & \pi_{4,10}, & \pi_{4,11}, & \pi_{4,12} \\ \pi_{5,7}, & \pi_{5,8}, & \pi_{5,9}, & \pi_{5,10}, & \pi_{5,11}, & \pi_{5,12} \\ \pi_{6,7}, & \pi_{6,8}, & \pi_{6,9}, & \pi_{6,10}, & \pi_{6,11}, & \pi_{6,12} \end{vmatrix} = 0,$$

i.e.,

$$\Pi \begin{pmatrix} 1, 2, 3, 4, 5, 6 \\ 7, 8, 9, 10, 11, 12 \end{pmatrix} = 0 ; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (213)$$

a relation which is clearly true when any of the spheres are replaced by planes, or points, or the plane at infinity.

203. An important case is when the plane at infinity is a member of both systems of spheres; thus taking the two systems as $(\theta, 1, 2, 3, 4, 5)$, $(\theta, 6, 7, 8, 9, 10)$, we have

$$\Pi\begin{pmatrix} \theta, 1, 2, 3, 4, 5 \\ \theta, 6, 7, 8, 9, 10 \end{pmatrix} = 0 ;$$

whence, if the radii of the spheres be all different from zero, and they cut at angles $\omega_{1,6}$, &c., we have

$$\begin{vmatrix} 0, & \dfrac{1}{r_1}, & \dfrac{1}{r_2}, & \dfrac{1}{r_3}, & \dfrac{1}{r_4}, & \dfrac{1}{r_5} \\[2mm] \dfrac{1}{r_6}, & \cos\omega_{1,6}, & \cos\omega_{2,6}, & \cos\omega_{3,6}, & \cos\omega_{4,6}, & \cos\omega_{5,6} \\[2mm] \dfrac{1}{r_7}, & \cos\omega_{1,7}, & \cos\omega_{2,7}, & \cos\omega_{3,7}, & \cos\omega_{4,7}, & \cos\omega_{5,7} \\[2mm] \dfrac{1}{r_8}, & \cos\omega_{1,8}, & \cos\omega_{2,8}, & \cos\omega_{3,8}, & \cos\omega_{4,8}, & \cos\omega_{5,8} \\[2mm] \dfrac{1}{r_9}, & \cos\omega_{1,9}, & \cos\omega_{2,9}, & \cos\omega_{3,9}, & \cos\omega_{4,9}, & \cos\omega_{5,9} \\[2mm] \dfrac{1}{r_{10}}, & \cos\omega_{1,10}, & \cos\omega_{2,10}, & \cos\omega_{3,10}, & \cos\omega_{4,10}, & \cos\omega_{5,10} \end{vmatrix} = 0 . \quad (214)$$

204. If we have two systems of five spheres each, say $(1, 2, 3, 4, 5)$, $(6, 7, 8, 9, 10)$, then we have

$$\Pi\begin{pmatrix} 1, 2, 3, 4, 5 \\ 6, 7, 8, 9, 10 \end{pmatrix} = \begin{vmatrix} 1, & 2f_1, & 2g_1, & 2h_1, & c_1 \\ 1, & 2f_2, & 2g_2, & 2h_2, & c_2 \\ 1, & 2f_3, & 2g_3, & 2h_3, & c_3 \\ 1, & 2f_4, & 2g_4, & 2h_4, & c_4 \\ 1, & 2f_5, & 2g_5, & 2h_5, & c_5 \end{vmatrix} \times \begin{vmatrix} c_6, & -f_6, & -g_6, & -h_6, & 1 \\ c_7, & -f_7, & -g_7, & -h_7, & 1 \\ c_8, & -f_8, & -g_8, & -h_8, & 1 \\ c_9, & -f_9, & -g_9, & -h_9, & 1 \\ c_{10}, & -f_{10}, & -g_{10}, & -h_{10}, & 1 \end{vmatrix} ;$$

and hence we see that,

$$\left\{ \Pi\begin{pmatrix} 1, 2, 3, 4, 5 \\ 6, 7, 8, 9, 10 \end{pmatrix} \right\}^2 = \Pi\begin{pmatrix} 1, 2, 3, 4, 5 \\ 1, 2, 3, 4, 5 \end{pmatrix} \times \Pi\begin{pmatrix} 6, 7, 8, 9, 10 \\ 6, 7, 8, 9, 10 \end{pmatrix} \quad . \quad . \quad . \quad (215)$$

205. Again we have

$$\Pi\begin{pmatrix} \theta, 1, 2, 3, 4 \\ \theta, 1, 2, 3, 4 \end{pmatrix} = \begin{vmatrix} 0, & 0, & 0, & 0, & 1 \\ 1, & 2f_1, & 2g_1, & 2h_1, & c_1 \\ 1, & 2f_2, & 2g_2, & 2h_2, & c_2 \\ 1, & 2f_3, & 2g_3, & 2h_3, & c_3 \\ 1, & 2f_4, & 2g_4, & 2h_4, & c_4 \end{vmatrix} \times \begin{vmatrix} 1, & 0, & 0, & 0, & 0 \\ c_1, & -f_1, & -g_1, & -h_1, & 1 \\ c_2, & -f_2, & -g_2, & -h_2, & 1 \\ c_3, & -f_3, & -g_3, & -h_3, & 1 \\ c_4, & -f_4, & -g_4, & -h_4, & 1 \end{vmatrix}$$

$$= 8 \begin{vmatrix} -f_1, & -g_1, & -h_1, & 1 \\ -f_2, & -g_2, & -h_2, & 1 \\ -f_3, & -g_3, & -h_3, & 1 \\ -f_4, & -g_4, & -h_4, & 1 \end{vmatrix}^2$$

$$= 288.\{V(1, 2, 3, 4)\}^2 ; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (216)$$

where $V(1, 2, 3, 4)$ denotes the volume of the tetrahedron, whose vertices are the centres of the spheres $(1, 2, 3, 4)$.

Again, let P be the common point of the spheres $(1, 2, 3)$, then if P be denoted by the symbol (4) we have

$$288.\{V(1, 2, 3, P)\}^2 = \Pi\begin{pmatrix} \theta, 1, 2, 3, 4 \\ \theta, 1, 2, 3, 4 \end{pmatrix}$$

$$= \begin{vmatrix} 0, & 1, & 1, & 1, & 1 \\ 1, & \pi_{1,1}, & \pi_{1,2}, & \pi_{1,3}, & 0 \\ 1, & \pi_{2,1}, & \pi_{2,2}, & \pi_{2,3}, & 0 \\ 1, & \pi_{3,1}, & \pi_{3,3}, & \pi_{3,4}, & 0 \\ 1, & 0, & 0, & 0, & 0 \end{vmatrix}$$

$$= -\Pi\begin{pmatrix} 1, 2, 3 \\ 1, 2, 3 \end{pmatrix}.$$

Thus, if P be a common point of the system $(1, 2, 3)$,

$$\Pi\begin{pmatrix} 1, 2, 3 \\ 1, 2, 3 \end{pmatrix} = -288.\{V(1, 2, 3, P)\}^2. \quad . \quad . \quad . \quad . \quad . \quad (217)$$

CHAPTER II.—SPECIAL SYSTEMS OF SPHERES.

Sphere Cutting Four given Spheres Orthogonally.—§§ 206–208.

206. Let x denote the sphere which cuts the four given spheres (1, 2, 3, 4) orthogonally ; then, since

$$\Pi\begin{pmatrix} \theta, & x, & 1, & 2, & 3, & 4 \\ \theta, & x, & 1, & 2, & 3, & 4 \end{pmatrix} = 0,$$

we have

$$\pi_{x,x}.\Pi\begin{pmatrix} \theta, & 1, & 2, & 3, & 4 \\ \theta, & 1, & 2, & 3, & 4 \end{pmatrix} = \Pi\begin{pmatrix} 1, & 2, & 3, & 4 \\ 1, & 2, & 3, & 4 \end{pmatrix} ;$$

whence, by equation (216),

$$\pi_{x,x} = \frac{\Pi\begin{pmatrix} 1, & 2, & 3, & 4 \\ 1, & 2, & 3, & 4 \end{pmatrix}}{288.\{V(1, 2, 3, 4)\}^2}.$$

Hence the radius of the sphere is given by

$$r = \frac{\left\{ -\Pi\begin{pmatrix} 1, & 2, & 3, & 4 \\ 1, & 2, & 3, & 4 \end{pmatrix} \right\}^{\frac{1}{2}}}{24.V(1, 2, 3, 4)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (218)$$

207. If the radii of the spheres (1, 2, 3, 4) be all zero, and the sides of the tetrahedron (1, 2, 3, 4) be denoted by a, b, c, a', b', c', we have at once,

$$-\Pi\begin{pmatrix} 1, & 2, & 3, & 4 \\ 1, & 2, & 3, & 4 \end{pmatrix} = - \begin{vmatrix} 0, & a^2, & b^2, & c^2 \\ a^2, & 0, & c'^2, & b'^2 \\ b^2, & c'^2, & 0, & a'^2 \\ c^2, & b'^2, & a'^2, & 0 \end{vmatrix}$$

$$= 2b^2b'^2c^2c'^2 + 2c^2c'^2a^2a'^2 + 2a^2a'^2b^2b'^2 - a^4a'^4 - b^4b'^4 - c^4c'^4$$

$$= 16\ \sigma.(\sigma - aa')(\sigma - bb')(\sigma - cc') ;$$

where

$$2\sigma = aa' + bb' + cc'.$$

Hence the radius of the sphere circumscribing a tetrahedron is equal to

$$\frac{1}{6} \frac{\{\sigma(\sigma - aa')(\sigma - bb')(\sigma - cc')\}^{\frac{1}{2}}}{V}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (219)$$

where V denotes the volume of the tetrahedron, which agrees with the known value (TODHUNTER, ' Spherical Trig.,' § 163).

208. If the four spheres meet in a point, the sphere which cuts them orthogonally will be coincident with this point, and so the radius must be zero. Hence, if the system (1, 2, 3, 4) have a common point, we must have

$$\Pi\begin{pmatrix}1, 2, 3, 4\\1, 2, 3, 4\end{pmatrix}=0. \quad\quad \text{. . . . . (220)}$$

Five Spheres having a Common Orthogonal Sphere.—§§ 209–211.

209. Suppose the system of spheres (1, 2, 3, 4, 5) have a common orthogonal sphere, x say; then, y denoting any other sphere, the equation

$$\Pi\begin{pmatrix}x, 1, 2, 3, 4, 5\\y, 1, 2, 3, 4, 5\end{pmatrix}=0,$$

leads at once to the condition

$$\Pi\begin{pmatrix}1, 2, 3, 4, 5\\1, 2, 3, 4, 5\end{pmatrix}=0; \quad\quad \text{. . . . . (221)}$$

which is the necessary and sufficient condition that the system may have a common orthogonal sphere.

Similarly, if (6, 7, 8, 9, 10) denote any other system of spheres, we should have, since

$$\Pi\begin{pmatrix}x, 6, 7, 8, 9, 10\\y, 1, 2, 3, 4, 5\end{pmatrix}=0,$$

$$\Pi\begin{pmatrix}6, 7, 8, 9, 10\\1, 2, 3, 4, 5\end{pmatrix}=0;$$

and hence

$$\left\{\Pi\begin{pmatrix}6, 7, 8, 9, 10\\1, 2, 3, 4, 5\end{pmatrix}\right\}^{2}=\Pi\begin{pmatrix}1, 2, 3, 4, 5\\1, 2, 3, 4, 5\end{pmatrix}\times\Pi\begin{pmatrix}6, 7, 8, 9, 10\\6, 7, 8, 9, 10\end{pmatrix}.$$

210. It is easy to prove, that if the system of spheres (1, 2, 3, 4, 5) be such that the condition (221) is satisfied, then any four of them will be connected with any four other spheres (6, 7, 8, 9) by the relation

$$\left\{\Pi\begin{pmatrix}6, 7, 8, 9\\1, 2, 3, 4\end{pmatrix}\right\}^{2}=\Pi\begin{pmatrix}1, 2, 3, 4\\1, 2, 3, 4\end{pmatrix}\times\Pi\begin{pmatrix}6, 7, 8, 9\\6, 7, 8, 9\end{pmatrix}.$$

211 Suppose now (x) to denote any sphere, then the equation

$$\Pi\begin{pmatrix}x, 1, 2, 3, 4\\1, 2, 3, 4, 5\end{pmatrix}=0$$

gives us

$$\pi_{x,1}.\Pi\begin{pmatrix}1, 2, 3, 4\\2, 3, 4, 5\end{pmatrix}-\pi_{x,2}.\Pi\begin{pmatrix}1, 2, 3, 4\\1, 3, 4, 5\end{pmatrix}+\pi_{x,3}.\Pi\begin{pmatrix}1, 2, 3, 4\\1, 2, 4, 5\end{pmatrix}$$

$$-\pi_{x,4}.\Pi\begin{pmatrix}1, 2, 3, 4\\1, 2, 3, 5\end{pmatrix}+\pi_{x,5}.\Pi\begin{pmatrix}1, 2, 3, 4\\1, 2, 3, 4\end{pmatrix}=0 ;$$

applying the theorem of § 210, we have

$$\pi_{x,5}.\left\{\Pi\begin{pmatrix}1, 2, 3, 4\\1, 2, 3, 4\end{pmatrix}\right\}^{\frac{1}{2}}=\pi_{x,1}.\left\{\Pi\begin{pmatrix}2, 3, 4, 5\\2, 3, 4, 5\end{pmatrix}\right\}^{\frac{1}{2}}+\pi_{x,2}.\left\{\Pi\begin{pmatrix}3, 4, 5, 1\\3, 4, 5, 1\end{pmatrix}\right\}^{\frac{1}{2}}$$

$$+\pi_{x,3}.\left\{\Pi\begin{pmatrix}4, 5, 1, 2\\4, 5, 1, 2\end{pmatrix}\right\}^{\frac{1}{2}}+\pi_{x,4}.\left\{\Pi\begin{pmatrix}5, 1, 2, 3\\5, 1, 2, 3\end{pmatrix}\right\}^{\frac{1}{2}}.$$

But if r denote the radius of the common orthogonal sphere, we have by equation (218)

$$-576r^2=\frac{\Pi\begin{pmatrix}1, 2, 3, 4\\1, 2, 3, 4\end{pmatrix}}{V(1, 2, 3, 4)}=\frac{\Pi\begin{pmatrix}2, 3, 4, 5\\2, 3, 4, 5\end{pmatrix}}{V(2, 3, 4, 5)}=\frac{\Pi\begin{pmatrix}3, 4, 5, 1\\3, 4, 5, 1\end{pmatrix}}{V(3, 4, 5, 1)}=\frac{\Pi\begin{pmatrix}4, 5, 1, 2\\4, 5, 1, 2\end{pmatrix}}{V(4, 5, 1, 2)}=\frac{\Pi\begin{pmatrix}5, 1, 2, 3\\5, 1, 2, 3\end{pmatrix}}{V(5, 1, 2, 3)} ;$$

and thus our equation becomes

$$\pi_{x,5}.V(1,\ 2,\ 3,\ 4)=\pi_{x,1}.V(2,\ 3,\ 4,\ 5)+\pi_{x,2}.V(3,\ 4,\ 5,\ 1)$$

$$+\pi_{x,3}.V(4,\ 5,\ 1,\ 2)+\pi_{x,4}.V(5,\ 1,\ 2,\ 3).$$

Thus, if any five spheres have a common orthogonal sphere, and the tetrahedral coordinates of the centre of one of them, (5) say, referred to the tetrahedron formed by the centres of the other four (1, 2, 3, 4), be $\alpha, \beta, \gamma, \delta$, then the powers of any other sphere are connected by the relation

$$\pi_{x,5}=\alpha.\pi_{x,1}+\beta.\pi_{x,2}+\gamma.\pi_{x,3}+\delta.\pi_{x,4}. \quad . \quad . \quad . \quad . \quad . \quad (222)$$

As a particular case, if A, B, C, D be the centres of (1, 2, 3, 4), P any point on the sphere cutting these orthogonally, and O be any other point,

$$OP^2=\alpha.(OA^2-r_1^2)+\beta.(OB^2-r_2^2)+\gamma.(OC^2-r_3^2)+\delta.(OD^2-r_4^2),$$

$\alpha, \beta, \gamma, \delta$ being the tetrahedral coordinates of P referred to ABCD.

Orthogonal Systems of Spheres.—§§ 212–214.

212. Five spheres may be said to form an orthogonal system if they cut one another orthogonally. It is clear that the centres of any four must form a tetrahedron, such that the perpendiculars from the angular points on the opposite faces meet in a point, viz., the centre of the fifth.

If the system be denoted by (1, 2, 3, 4, 5), then (x, y) denoting any other spheres, we have, since

$$\Pi\begin{pmatrix} x, & 1, & 2, & 3, & 4, & 5 \\ y, & 1, & 2, & 3, & 4, & 5 \end{pmatrix} = 0,$$

the formula

$$\pi_{x,y} = \frac{\pi_{x,1} \cdot \pi_{y,1}}{\pi_{1,1}} + \frac{\pi_{x,2} \cdot \pi_{y,2}}{\pi_{2,2}} + \frac{\pi_{x,3} \pi_{y,3}}{\pi_{3,3}} + \frac{\pi_{x,4} \cdot \pi_{y,4}}{\pi_{4,4}} + \frac{\pi_{x,5} \pi_{y,5}}{\pi_{5,5}} \; ;$$

or if the radii of the spheres be r_1, r_2, r_3, r_4, r_5, we have,

$$-2\pi_{x,y} = \frac{\pi_{x,1} \cdot \pi_{y,1}}{r_1^2} + \frac{\pi_{x,2} \cdot \pi_{y,2}}{r_2^2} + \frac{\pi_{x,3} \cdot \pi_{y,3}}{r_3^2} + \frac{\pi_{x,4} \pi_{y,4}}{r_4^2} + \frac{\pi_{x,5} \pi_{y,5}}{r_5^2} \cdot \quad \cdots \quad (223)$$

From this we can deduce at once the formulæ—

$$\left. \begin{aligned} -2 &= \frac{\pi_{x,1}}{r_1^2} + \frac{\pi_{x,2}}{r_2^2} + \frac{\pi_{x,3}}{r_3^2} + \frac{\pi_{x,4}}{r_4^2} + \frac{\pi_{x,5}}{r_5^2} \\ 4r_x^2 &= \frac{\pi_{x,1}^2}{r_1^2} + \frac{\pi_{x,2}^2}{r_2^2} + \frac{\pi_{x,3}^2}{r_3^2} + \frac{\pi_{x,4}^2}{r_4^2} + \frac{\pi_{x,5}^2}{r_5^2} \end{aligned} \right\} ; \quad \cdots \quad (224)$$

where x denotes any sphere.

Also

$$\left. \begin{aligned} 0 &= \frac{\pi_{x,1}}{r_1^2} + \frac{\pi_{x,2}}{r_2^2} + \frac{\pi_{x,3}}{r_3^2} + \frac{\pi_{x,4}}{r_4^2} + \frac{\pi_{x,5}}{r_5^2} \\ 4 &= \frac{\pi_{x,1}^2}{r_1^2} + \frac{\pi_{x,2}^2}{r_2^2} + \frac{\pi_{x,3}^2}{r_3^2} + \frac{\pi_{x,4}^2}{r_4^2} + \frac{\pi_{x,5}^2}{r_5^2} \end{aligned} \right\} , \quad \cdots \quad (225)$$

where x denotes any plane.

Again, taking x and y as coinciding with the plane at infinity, we have, since $\pi_{\theta,\theta} = 0$,

$$0 = \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} + \frac{1}{r_4^2} + \frac{1}{r_5^2} \; ; \quad \cdots \cdots \cdots \quad (226)$$

whence we see that one of the five spheres is imaginary.

213. If any system of spheres, say (1, 2, 3, 4, 5), be given, then the five spheres, each of which is orthogonal to four of the given system, form a system, say (6, 7, 8, 9, 10), which may be called the "orthogonal" system of the former.

If (x, y) denote any other spheres, the equation

$$\Pi\begin{pmatrix} x, & 1, & 2, & 3, & 4, & 5 \\ y, & 6, & 7, & 8, & 9, & 10 \end{pmatrix} = 0$$

gives us

$$
\begin{vmatrix}
\pi_{x,y}, & \pi_{x,6}, & \pi_{x,7}, & \pi_{x,8}, & \pi_{x,9}, & \pi_{x,10} \\
\pi_{y,1}, & \pi_{1,6}, & 0, & 0, & 0, & 0 \\
\pi_{y,2}, & 0, & \pi_{2,7}, & 0, & 0, & 0 \\
\pi_{y,3}, & 0, & 0, & \pi_{3,8}, & 0, & 0 \\
\pi_{y,4}, & 0, & 0, & 0, & \pi_{4,9}, & 0 \\
\pi_{y,5}, & 0, & 0, & 0, & 0, & \pi_{5,10}
\end{vmatrix} = 0 \, ;
$$

which may be written

$$
\pi_{x,y} = \frac{\pi_{x,6}\cdot\pi_{y,1}}{\pi_{1,6}} + \frac{\pi_{x,7}\cdot\pi_{y,2}}{\pi_{2,7}} + \frac{\pi_{x,8}\cdot\pi_{y,3}}{\pi_{3,8}} + \frac{\pi_{x,9}\pi_{y,4}}{\pi_{4,9}} + \frac{\pi_{x,10}\cdot\pi_{y,5}}{\pi_{5,10}} \cdot \quad . \quad . \quad . \quad . \quad (227)
$$

Hence x denoting any sphere, radius r_x,

$$
\left.
\begin{aligned}
1 &= \frac{\pi_{x,6}}{\pi_{1,6}} + \frac{\pi_{x,7}}{\pi_{2,7}} + \frac{\pi_{x,8}}{\pi_{3,8}} + \frac{\pi_{x,9}}{\pi_{4,9}} + \frac{\pi_{x,10}}{\pi_{5,10}} \\
-2r_x^2 &= \frac{\pi_{x,6}\cdot\pi_{x,1}}{\pi_{1,6}} + \frac{\pi_{x,7}\cdot\pi_{x,2}}{\pi_{2,7}} + \frac{\pi_{x,8}\cdot\pi_{x,3}}{\pi_{3,8}} + \frac{\pi_{x,9}\cdot\pi_{x,4}}{\pi_{4,9}} + \frac{\pi_{x,10}\cdot\pi_{x,5}}{\pi_{5,10}}
\end{aligned}
\right\} \quad . \quad . \quad . \quad (228)
$$

Or again, if x denote any plane,

$$
\left.
\begin{aligned}
0 &= \frac{\pi_{x,6}}{\pi_{1,6}} + \frac{\pi_{x,7}}{\pi_{2,7}} + \frac{\pi_{x,8}}{\pi_{3,8}} + \frac{\pi_{x,9}}{\pi_{4,9}} + \frac{\pi_{x,10}}{\pi_{5,10}} \\
-2 &= \frac{\pi_{x,6}\cdot\pi_{x,1}}{\pi_{1,6}} + \frac{\pi_{x,7}\cdot\pi_{x,2}}{\pi_{2,7}} + \frac{\pi_{x,8}\cdot\pi_{x,3}}{\pi_{3,8}} + \frac{\pi_{x,9}\cdot\pi_{x,4}}{\pi_{4,9}} + \frac{\pi_{x,10}\cdot\pi_{x,5}}{\pi_{5,10}}
\end{aligned}
\right\} \quad . \quad . \quad . \quad (229)
$$

Also, taking x and y as coinciding with the plane at infinity,

$$
0 = \frac{1}{\pi_{1,6}} + \frac{1}{\pi_{2,7}} + \frac{1}{\pi_{3,8}} + \frac{1}{\pi_{4,9}} + \frac{1}{\pi_{5,10}} \cdot \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (230)
$$

A particular case would be any five points in space, and the five spheres circumscribing the five tetrahedra; thus by equation (230) we see that the sum of the squares of the reciprocals of the tangents from each of five points to the spheres passing through the remaining four is zero.

214. A system of three spheres and their two points of intersection constitute an important system, which may be called a "semi-orthogonal" system. Denoting the three spheres by (1, 2, 3), and their points of intersection by (4, 5), then if (x, y) denote any other spheres, the equation

$$\Pi\begin{pmatrix} x, 1, 2, 3, 4, 5 \\ y, 1, 2, 3, 4, 5 \end{pmatrix} = 0$$

becomes

$$\begin{vmatrix} \pi_{x,y}, & \pi_{x,1}, & \pi_{x,2}, & \pi_{x,3}, & \pi_{x,4}, & \pi_{x,5} \\ \pi_{y,1}, & \pi_{1,1}, & 0, & 0, & 0, & 0 \\ \pi_{y,2}, & 0, & \pi_{2,2}, & 0, & 0, & 0 \\ \pi_{y,3}, & 0, & 0, & \pi_{3,3}, & 0, & 0 \\ \pi_{y,4}, & 0, & 0, & 0, & 0, & \pi_{4,5} \\ \pi_{y,5}, & 0, & 0, & 0, & \pi_{4,5}, & 0 \end{vmatrix} = 0 \; ;$$

which may be written

$$\pi_{x,y} = \frac{\pi_{x,1}.\pi_{y,1}}{\pi_{1,1}} + \frac{\pi_{x,2}.\pi_{y,2}}{\pi_{2,2}} + \frac{\pi_{x,3}.\pi_{y,3}}{\pi_{3,3}} + \frac{\pi_{x,4}\pi_{y,5} + \pi_{x,5}\pi_{y,4}}{\pi_{4,5}}. \quad \ldots \quad (231)$$

Denoting the radii of the spheres by r_1, r_2, r_3, and the distance between their common points by e, we have—

if x denote any sphere, radius r_x,

$$\left. \begin{aligned} -2 &= \frac{\pi_{x,1}}{r_1^2} + \frac{\pi_{x,2}}{r_2^2} + \frac{\pi_{x,3}}{r_3^2} - 2\frac{\pi_{x,4} + \pi_{x,5}}{e^2} \\ 4r_x^2 &= \frac{\pi_{x,1}^2}{r_1^2} + \frac{\pi_{x,2}^2}{r_2^2} + \frac{\pi_{x,3}^2}{r_3^2} - 4\frac{\pi_{x,4}.\pi_{x,5}}{e^2} \end{aligned} \right\} \; ; \quad \ldots \quad (232)$$

if x denote any plane,

$$\left. \begin{aligned} 0 &= \frac{\pi_{x,1}}{r_1^2} + \frac{\pi_{x,2}}{r_2^2} + \frac{\pi_{x,3}}{r_3^2} - 2\frac{\pi_{x,4} + \pi_{x,5}}{e^2} \\ 4 &= \frac{\pi_{x,1}^2}{r_1^2} + \frac{\pi_{x,2}^2}{r_2^2} + \frac{\pi_{x,3}^2}{r_3^2} - 4\frac{\pi_{x,4}.\pi_{x,5}}{e^2} \end{aligned} \right\} \quad \ldots \quad (233)$$

Also, taking x and y as coincident with the plane at infinity, (231) becomes

$$0 = \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} - \frac{4}{e^2}. \quad \ldots \quad (234)$$

Spheres touching one another.—§§ 215, 216.

215. If the system of spheres (1, 2, 3, 4, 5) touch each other externally, the equation

$$\Pi\begin{pmatrix} \theta, 1, 2, 3, 4, 5 \\ \theta, 1, 2, 3, 4, 5 \end{pmatrix} = 0,$$

becomes

$$\begin{vmatrix} 0, & \dfrac{1}{r_1}, & \dfrac{1}{r_2}, & \dfrac{1}{r_3}, & \dfrac{1}{r_4}, & \dfrac{1}{r_5} \\[2mm] \dfrac{1}{r_1}, & -1, & 1, & 1, & 1, & 1 \\[2mm] \dfrac{1}{r_2}, & 1, & -1, & 1, & 1, & 1 \\[2mm] \dfrac{1}{r_3}, & 1, & 1, & -1, & 1, & 1 \\[2mm] \dfrac{1}{r_4}, & 1, & 1, & 1, & -1, & 1 \\[2mm] \dfrac{1}{r_5}, & 1, & 1, & 1, & 1, & -1 \end{vmatrix} = 0 ;$$

which may be written

$$\Sigma \frac{1}{r^2} = \Sigma \frac{1}{r_1 r_2} ; \quad \cdots \cdots \cdots \quad (235)$$

or

$$3\left(\frac{1}{r_1{}^2} + \frac{1}{r_2{}^2} + \frac{1}{r_3{}^2} + \frac{1}{r_4{}^2} + \frac{1}{r_5{}^2} \right) = \left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} + \frac{1}{r_5} \right)^2 .$$

216. If the spheres $(1, 2, 3, 4, 5)$ have a common tangent sphere, which we will denote by (x), then the equation

$$\Pi \begin{pmatrix} x, & 1, 2, 3, 4, 5 \\ x, & 1, 2, 3, 4, 5 \end{pmatrix} = 0,$$

gives us

$$\begin{vmatrix} 1, & \sqrt{\pi_{1,1}}, & \sqrt{\pi_{2,2}}, & \sqrt{\pi_{3,3}}, & \sqrt{\pi_{4,4}}, & \sqrt{\pi_{5,5}} \\[2mm] \sqrt{\pi_{1,1}}, & \pi_{1,1}, & \pi_{1,2}, & \pi_{1,3}, & \pi_{1,4}, & \pi_{1,5} \\[2mm] \sqrt{\pi_{2,2}}, & \pi_{2,1}, & \pi_{2,2}, & \pi_{2,3}, & \pi_{2,4}, & \pi_{2,5} \\[2mm] \sqrt{\pi_{3,3}}, & \pi_{3,1}, & \pi_{3,2}, & \pi_{3,3}, & \pi_{3,4}, & \pi_{3,5} \\[2mm] \sqrt{\pi_{4,4}}, & \pi_{4,1}, & \pi_{4,2}, & \pi_{4,3}, & \pi_{4,4}, & \pi_{4,5} \\[2mm] \sqrt{\pi_{5,5}}, & \pi_{5,1}, & \pi_{5,2}, & \pi_{5,3}, & \pi_{5,4}, & \pi_{5,5} \end{vmatrix} = 0 ;$$

where the positive or negative sign is to be taken with any expression, such as $\sqrt{\pi_{m,n}}$, according as the spheres (m, n) have contact with (x) of the same or opposite kind.

Now if we write $\pi_{r,s} \pm \sqrt{\pi_{r,r} \cdot \pi_{s,s}} = t_{r,s}{}^2$, in which $t_{r,s}$ will denote the length of the common tangent of the two spheres (r, s), " direct " if the positive sign is to be taken, " indirect " if the negative sign is to be taken ; then we can deduce at once from this equation the equation of condition, due to Dr. CASEY,

$$\begin{vmatrix} 0, & t_{1,2}{}^2, & t_{1,3}{}^2, & t_{1,4}{}^2, & t_{1,5}{}^2 \\ t_{2,1}{}^2, & 0, & t_{2,3}{}^2, & t_{2,4}{}^2, & t_{2,5}{}^2 \\ t_{3,1}{}^2, & t_{3,2}{}^2, & 0, & t_{3,4}{}^2, & t_{3,5}{}^2 \\ t_{4,1}{}^2, & t_{4,2}{}^2, & t_{4,3}{}^2, & 0, & t_{4,5}{}^2 \\ t_{5,1}{}^2, & t_{5,2}{}^2, & t_{5,3}{}^2, & t_{5,4}{}^2, & 0 \end{vmatrix} = 0, \quad \ldots \ldots \quad (236)$$

which must be satisfied if the spheres all touch the same sphere. Supposing this condition satisfied, the radius of the tangent sphere is easily found thus. Let (6, 7, 8, 9, 10) denote the system of spheres orthogonal to the spheres (1, 2, 3, 4, 5), then by equation (228) we have

$$\frac{\pi_{x,1}}{\pi_{1,6}} + \frac{\pi_{x,2}}{\pi_{2,7}} + \frac{\pi_{x,3}}{\pi_{3,8}} + \frac{\pi_{x,4}}{\pi_{4,9}} + \frac{\pi_{x,5}}{\pi_{5,10}} = 1 ;$$

hence

$$\frac{1}{r_x} = 2 \left\{ \frac{r_1}{\pi_{1,6}} + \frac{r_2}{\pi_{2,7}} + \frac{r_3}{\pi_{3,8}} + \frac{r_4}{\pi_{4,9}} + \frac{r_5}{\pi_{5,10}} \right\}, \quad \ldots \ldots \quad (237)$$

where the radii r_1, r_2, &c. are to be taken with the positive or negative sign, according as the contact is external or internal.

CHAPTER III.—SPHERES CONNECTED WITH A SYSTEM OF FOUR SPHERES.

In this chapter it is proposed to extend a few of the results arrived at in Chapter IV., Part I. The formulæ for the spheres which pass through the points of intersection of four spheres, and also for the spheres which touch four spheres, will be seen to be exactly analogous to those proved for circles passing through the points of intersection of, and touching three given circles. In the case of a tetrahedron there does not seem to be any sphere analogous to the nine-points circle of a triangle; a special system of spheres will, however, be mentioned, which have a sphere touching their tangent spheres, but even here there is no formula, connecting the radius of this sphere with that of the corresponding sphere circumscribing the tetrahedron formed by the four spheres, analogous to the formula $(2\rho = R)$, connecting the radius of the nine-points circle of a triangle with its circumcircle.

Spheres cutting Four Spheres at given Angles.—§§ 217–219.

217. Let the given spheres be denoted by (1, 2, 3, 4), and let their orthogonal sphere be denoted by the symbol (5). Let x denote any sphere cutting (1, 2, 3, 4) at the angles θ, ϕ, ψ, χ; and let x cut (5) at the angle ω.

From the equation

$$\Pi\begin{pmatrix} x, & 1, & 2, & 3, & 4, & 5 \\ x, & 1, & 2, & 3, & 4, & 5 \end{pmatrix}=0,$$

we may deduce

$$\begin{vmatrix} -1, & \cos\theta, & \cos\phi, & \cos\psi, & \cos\chi, & \cos\omega \\ \cos\theta, & -1, & \cos\omega_{1,2}, & \cos\omega_{1,3}, & \cos\omega_{1,4}, & 0 \\ \cos\phi, & \cos\omega_{2,1}, & -1, & \cos\omega_{2,3}, & \cos\omega_{2,4}, & 0 \\ \cos\psi, & \cos\omega_{3,1}, & \cos\omega_{3,2}, & -1, & \cos\omega_{3,4}, & 0 \\ \cos\chi, & \cos\omega_{4,1}, & \cos\omega_{4,2}, & \cos\omega_{4,3}, & -1, & 0 \\ \cos\omega, & 0, & 0, & 0, & 0, & -1 \end{vmatrix}=0 ;$$

whence we have

$$\sin^2\omega \begin{vmatrix} -1, & \cos\omega_{1,2}, & \cos\omega_{1,3}, & \cos\omega_{1,4} \\ \cos\omega_{2,1}, & -1, & \cos\omega_{2,3}, & \cos\omega_{2,4} \\ \cos\omega_{3,1}, & \cos\omega_{3,2}, & -1, & \cos\omega_{3,4} \\ \cos\omega_{4,1}, & \cos\omega_{4,2}, & \cos\omega_{4,3}, & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 0, & \cos\theta, & \cos\phi, & \cos\psi, & \cos\chi \\ \cos\theta, & -1, & \cos\omega_{1,2}, & \cos\omega_{1,3}, & \cos\omega_{1,4} \\ \cos\phi, & \cos\omega_{2,1}, & -1, & \cos\omega_{2,3}, & \cos\omega_{2,4} \\ \cos\psi, & \cos\omega_{3,1}, & \cos\omega_{3,2}, & -1, & \cos\omega_{3,4} \\ \cos\chi, & \cos\omega_{4,1}, & \cos\omega_{4,2}, & \cos\omega_{4,3}, & -1 \end{vmatrix} \quad . \quad . \quad . \quad . \quad (238)$$

Let ρ denote the radius of the sphere (x), then by the equation

$$\Pi\begin{pmatrix} x, & 1, & 2, & 3, & 4, & 5 \\ \theta, & 1, & 2, & 3, & 4, & 5 \end{pmatrix}=0,$$

we have

$$\begin{vmatrix} \dfrac{1}{\rho}, & \dfrac{1}{r_1}, & \dfrac{1}{r_2}, & \dfrac{1}{r_3}, & \dfrac{1}{r_4}, & \dfrac{1}{r_5} \\ \cos\theta, & -1, & \cos\omega_{1,2}, & \cos\omega_{1,3}, & \cos\omega_{1,4}, & 0 \\ \cos\phi, & \cos\omega_{2,1}, & -1, & \cos\omega_{2,3}, & \cos\omega_{2,4}, & 0 \\ \cos\psi, & \cos\omega_{3,1}, & \cos\omega_{3,2}, & -1, & \cos\omega_{3,4}, & 0 \\ \cos\chi, & \cos\omega_{4,1}, & \cos\omega_{4,2}, & \cos\omega_{4,3}, & -1, & 0 \\ \cos\omega, & 0, & 0, & 0, & 0, & -1 \end{vmatrix}=0 ;$$

or

$$\left(\frac{1}{\rho}+\frac{\cos \omega}{r_5}\right)
\begin{vmatrix}
-1, & \cos \omega_{1,2}, & \cos \omega_{1,3}, & \cos \omega_{1,4} \\
\cos \omega_{2,1}, & -1, & \cos \omega_{2,3}, & \cos \omega_{2,4} \\
\cos \omega_{3,1}, & \cos \omega_{3,2}, & -1, & \cos \omega_{3,4} \\
\cos \omega_{4,1}, & \cos \omega_{4,2}, & \cos \omega_{4,3}, & -1
\end{vmatrix}$$

$$+
\begin{vmatrix}
0, & \dfrac{1}{r_1}, & \dfrac{1}{r_2}, & \dfrac{1}{r_3}, & \dfrac{1}{r_4} \\
\cos \theta, & -1, & \cos \omega_{1,2}, & \cos \omega_{1,3}, & \cos \omega_{1,4} \\
\cos \phi, & \cos \omega_{2,1}, & -1, & \cos \omega_{2,3}, & \cos \omega_{2,4} \\
\cos \psi, & \cos \omega_{3,1}, & \cos \omega_{3,2}, & -1, & \cos \omega_{3,4} \\
\cos \chi, & \cos \omega_{4,1}, & \cos \omega_{4,2}, & \cos \omega_{4,3}, & -1
\end{vmatrix}
=0. \quad . \quad . \quad (239)$$

We infer, then, that two spheres can be drawn to cut the given spheres at the angles $(\theta, \phi, \psi, \chi)$, or else at angles supplementary to them. If the two spheres cut (1, 2, 3, 4) at the same angles, they cut the orthogonal sphere at supplementary angles, and *vice versâ*; and evidently one is the inverse of the other with respect to the orthogonal sphere.

Denoting their radii by ρ, ρ' we have

$$\frac{1}{\rho}-\frac{1}{\rho'}+\frac{2 \cos \omega}{r_5}=0,$$

and

$$\frac{1}{\rho}+\frac{1}{\rho'}=\text{F}.\cos \theta+\text{G}.\cos \phi+\text{H}.\cos \psi+\text{K}.\cos \chi, \quad . \quad . \quad . \quad . \quad (240)$$

where F, G, H, K are independent of $(\theta, \phi, \psi, \chi)$.

We see at once, then, that the two spheres will be real, coincident, or imaginary, according as $\dfrac{\cos^2 \omega}{r_5^2}$ is positive, zero, or negative.

But by equation (218) we see that the sign of r_5^2 is opposite to that of

$$\Pi\begin{pmatrix} 1, & 2, & 3, & 4 \\ 1, & 2, & 3, & 4 \end{pmatrix};$$

i.e., opposite to the sign of

$$\begin{vmatrix}
-1, & \cos \omega_{1,2}, & \cos \omega_{1,3}, & \cos \omega_{1,4} \\
\cos \omega_{2,1}, & -1, & \cos \omega_{2,3}, & \cos \omega_{2,4} \\
\cos \omega_{3,1}, & \cos \omega_{3,2}, & -1, & \cos \omega_{3,4} \\
\cos \omega_{4,1}, & \cos \omega_{4,2}, & \cos \omega_{4,3}, & -1
\end{vmatrix} \cdot$$

Hence, by equation (238), the two spheres which cut (1, 2, 3, 4) at the angles $(\theta, \phi, \psi, \chi)$ will be real, coincident, or imaginary, according as the expression

$$
\begin{vmatrix}
-1, & \cos\theta, & \cos\phi, & \cos\psi, & \cos\chi \\
\cos\theta, & -1, & \cos\omega_{1,2}, & \cos\omega_{1,3}, & \cos\omega_{1,4} \\
\cos\phi, & \cos\omega_{2,1}, & -1, & \cos\omega_{2,3}, & \cos\omega_{2,4} \\
\cos\psi, & \cos\omega_{3,1}, & \cos\omega_{3,2}, & -1, & \cos\omega_{3,4} \\
\cos\chi, & \cos\omega_{4,1}, & \cos\omega_{4,2}, & \cos\omega_{4,3}, & -1
\end{vmatrix}
$$

is positive, zero, or negative; *i.e.*, according as

$$
\Pi\begin{pmatrix} x, 1, 2, 3, 4 \\ x, 1, 2, 3, 4 \end{pmatrix}
$$

is positive, zero, or negative.

218. It is evident that eight pairs of spheres can be drawn to cut the four given spheres at angles whose cosines are $\pm\kappa_1$, $\pm\kappa_2$, $\pm\kappa_3$, $\pm\kappa_4$. If we denote the radii of these pairs by ρ_1, ρ'_1; ρ_2, ρ'_2; &c., we have by equation (240)

$$
\frac{1}{\rho}+\frac{1}{\rho'}=\pm\mathrm{F}.\kappa_1\pm\mathrm{G}.\kappa_2\pm\mathrm{H}.\kappa_3\pm\mathrm{K}.\kappa_4.
$$

Hence we have the relation

$$
\begin{Vmatrix}
\frac{1}{\rho_1}+\frac{1}{\rho'_1}, & \frac{1}{\rho_2}+\frac{1}{\rho'_2}, & \frac{1}{\rho_3}+\frac{1}{\rho'_3}, & \frac{1}{\rho_4}+\frac{1}{\rho'_4}, & \frac{1}{\rho_5}+\frac{1}{\rho'_5}, & \frac{1}{\rho_6}+\frac{1}{\rho'_6}, & \frac{1}{\rho_7}+\frac{1}{\rho'_7}, & \frac{1}{\rho_8}+\frac{1}{\rho'_8} \\
1, & -1, & 1, & 1, & 1, & -1, & 1, & 1 \\
1, & 1, & -1, & 1, & 1, & 1, & -1, & 1 \\
1, & 1, & 1, & -1, & 1, & 1, & 1, & -1 \\
1, & 1, & 1, & 1, & -1, & -1, & -1, & -1
\end{Vmatrix}=0. \quad . \quad (241)
$$

219. In the case of a tetrahedron formed by four planes, the orthogonal sphere becomes the plane at infinity; and so the pair of spheres which cut the faces at angles $(\theta\phi\psi\chi)$ coincide. Thus eight spheres can be drawn cutting the faces of a tetrahedron at angles equal or supplemental to $(\theta, \phi, \psi, \chi)$; and their radii are connected by the set of equations

$$
\begin{Vmatrix}
\frac{1}{\rho_1}, & \frac{1}{\rho_2}, & \frac{1}{\rho_3}, & \frac{1}{\rho_4}, & \frac{1}{\rho_5}, & \frac{1}{\rho_6}, & \frac{1}{\rho_7}, & \frac{1}{\rho_8} \\
1, & -1, & 1, & 1, & 1, & -1, & 1, & 1 \\
1, & 1, & -1, & 1, & 1, & 1, & -1, & 1 \\
1, & 1, & 1, & -1, & 1, & 1, & 1, & -1 \\
1, & 1, & 1, & 1, & -1, & -1, & -1, & -1
\end{Vmatrix}=0. \quad . \quad . \quad (242)
$$

A particular case is that of the eight tangent spheres, *i.e.*, one inscribed, four escribed to one face, three escribed to two faces, of a tetrahedron.

The Spheres Circumscribing the Tetrahedron Formed by Four Spheres.—§§ 220, 221.

220. Let the four given spheres be denoted by (1, 2, 3, 4), their orthogonal sphere by the symbol (5). It is evident that eight pairs of spheres can be drawn. Let P, Q, R, S be the four points in which they intersect, which lie within the tetrahedron formed by the centre of the spheres. Let x denote the sphere circumscribing PQRS.

We have by equation (220),

$$\Pi\begin{pmatrix} x, 2, 3, 4 \\ x, 2, 3, 4 \end{pmatrix} = \Pi\begin{pmatrix} x, 3, 4, 1 \\ x, 3, 4, 1 \end{pmatrix} = \Pi\begin{pmatrix} x, 4, 1, 2 \\ x, 4, 1, 2 \end{pmatrix} = \Pi\begin{pmatrix} x, 1, 2, 3 \\ x, 1, 2, 3 \end{pmatrix} = 0.$$

Hence, by a theorem of determinants,

$$\left.\begin{aligned}
\left\{\Pi\begin{pmatrix} x, 1, 2, 3 \\ x, 1, 2, 4 \end{pmatrix}\right\}^2 &= -\Pi\begin{pmatrix} x, 1, 2 \\ x, 1, 2 \end{pmatrix} \times \Pi\begin{pmatrix} x, 1, 2, 3, 4 \\ x, 1, 2, 3, 4 \end{pmatrix} \\
\left\{\Pi\begin{pmatrix} x, 2, 3, 4 \\ 1, 2, 3, 4 \end{pmatrix}\right\}^2 &= -\Pi\begin{pmatrix} 2, 3, 4 \\ 2, 3, 4 \end{pmatrix} \times \Pi\begin{pmatrix} x, 1, 2, 3, 4 \\ x, 1, 2, 3, 4 \end{pmatrix}
\end{aligned}\right\}.$$

But since

$$\Pi\begin{pmatrix} x, 1, 2, 3, 4, \theta \\ x, 1, 2, 3, 4, 5 \end{pmatrix} = 0,$$

we have

$$\pi_{x,5} \cdot \Pi\begin{pmatrix} \theta, 1, 2, 3, 4 \\ x, 1, 2, 3, 4 \end{pmatrix} - \Pi\begin{pmatrix} x, 1, 2, 3, 4 \\ x, 1, 2, 3, 4 \end{pmatrix} = 0,$$

or

$$\Pi\begin{pmatrix} x, 1, 2, 3, 4 \\ x, 1, 2, 3, 4 \end{pmatrix} = \pi_{x,5}\left\{\Pi\begin{pmatrix} 1, 2, 3, 4 \\ 1, 2, 3, 4 \end{pmatrix} - \Pi\begin{pmatrix} 1, 2, 3, 4 \\ x, 2, 3, 4 \end{pmatrix} - \Pi\begin{pmatrix} 1, 2, 3, 4 \\ 1, x, 3, 4 \end{pmatrix}\right.$$

$$\left. - \Pi\begin{pmatrix} 1, 2, 3, 4 \\ 1, 2, x, 4 \end{pmatrix} - \Pi\begin{pmatrix} 1, 2, 3, 4 \\ 1, 2, 3, x \end{pmatrix}\right\}. . \quad (243)$$

Also since

$$\Pi\begin{pmatrix} x, 1, 2, 3, 4, 5 \\ x, 1, 2, 3, 4, 5 \end{pmatrix} = 0,$$

we have

$$\pi_{5,5} \cdot \Pi\begin{pmatrix} x, 1, 2, 3, 4 \\ x, 1, 2, 3, 4 \end{pmatrix} = \pi^2_{x,5} \cdot \Pi\begin{pmatrix} 1, 2, 3, 4 \\ 1, 2, 3, 4 \end{pmatrix}. \quad . \quad . \quad . \quad . \quad (244)$$

But by equation (218),

$$\Pi\begin{pmatrix} 1, 2, 3, 4 \\ 1, 2, 3, 4 \end{pmatrix} = 288\pi_{5,5} \cdot \{V(1, 2, 3, 4)\}^2;$$

and by equation (217),

$$\Pi\begin{pmatrix} 1, 2, 3 \\ 1, 2, 3 \end{pmatrix} = -288 \cdot \{V(1, 2, 3, S)\}^2;$$

we deduce then from (243), that

$$\frac{\pi_{5,5}-\pi_{5,x}}{\pi_{5,x}}=\frac{V(P,2,3,4)+V(Q,3,4,1)+V(R,4,1,2)+V(S,1,2,3)}{V(1,2,3,4)};$$

i.e., denoting the radius of the sphere (P, Q, R, S) by ρ, and its angle of intersection with (5) by ω,

$$\frac{\rho\cos\omega}{r_5}=\frac{V(1,2,3,4)}{V(1,2,3,4)+V(P,2,3,4)+V(Q,3,4,1)+V(R,4,1,2)+V(S,1,2,3)}. \quad . \quad (245)$$

If P be without the tetrahedron formed by the centres of the spheres (1, 2, 3, 4) the sign of V(P, 2, 3, 4) must be changed, and thus the powers of the spheres with respect to the sphere orthogonal to (1, 2, 3, 4) can be written down.

Also we can easily deduce from (244), the equation

$$\begin{vmatrix} -\tan^2\omega, & \sqrt{\mu_{1,1}}, & \sqrt{\mu_{2,2}}, & \sqrt{\mu_{3,3}}, & \sqrt{\mu_{4,4}} \\ \sqrt{\mu_{1,1}}, & \mu_{1,1}, & \mu_{1,2}, & \mu_{1,3}, & \mu_{1,4} \\ \sqrt{\mu_{2,2}}, & \mu_{2,1}, & \mu_{2,2}, & \mu_{2,3}, & \mu_{2,4} \\ \sqrt{\mu_{3,3}}, & \mu_{3,1}, & \mu_{3,2}, & \mu_{3,3}, & \mu_{3,4} \\ \sqrt{\mu_{4,4}}, & \mu_{4,1}, & \mu_{4,2}, & \mu_{4,3}, & \mu_{4,4} \end{vmatrix}=0, \quad . \quad . \quad . \quad (246)$$

where $\mu_{1,1}$, $\mu_{1,2}$, &c. are the minors of $\pi_{1,1}$, $\pi_{1,2}$, &c. in the determinant $\Pi\begin{pmatrix}1,2,3,4\\1,2,3,4\end{pmatrix}$. Thus $\mu_{1,1}=-288\{V(P,2,3,4)\}^2$, and thus the sign of $\sqrt{\mu_{1,1}}$ is positive or negative according as P is within or without the tetrahedron formed by the centres of the spheres (1, 2, 3, 4).

221. The system of spheres (1, 2, 3, 4, x) is orthogonal to the system (P, Q, R, S, 5), and we get some interesting theorems by aid of § 213.

Thus, by equation (230), we have

$$\frac{1}{\pi_{P,1}}+\frac{1}{\pi_{Q,2}}+\frac{1}{\pi_{R,3}}+\frac{1}{\pi_{S,4}}+\frac{1}{\pi_{x,5}}=0. \quad . \quad . \quad . \quad . \quad . \quad (247)$$

And if x' denote the sphere passing through P', Q', R', S', the inverse points of P, Q, R, S with respect to (5), we have

$$\frac{1}{\pi_{P',1}}+\frac{1}{\pi_{Q',2}}+\frac{1}{\pi_{R',3}}+\frac{1}{\pi_{S',4}}+\frac{1}{\pi_{x',5}}=0.$$

But since x, x' are inverse with respect to (5),

$$\frac{1}{\pi_{x,5}}+\frac{1}{\pi_{x',5}}=\frac{2}{\pi_{5,5}};$$

hence

$$\frac{1}{\pi_{P,1}}+\frac{1}{\pi_{P',1}}+\frac{1}{\pi_{Q,2}}+\frac{1}{\pi_{Q',2}}+\frac{1}{\pi_{R,3}}+\frac{1}{\pi_{R',3}}+\frac{1}{\pi_{S,4}}+\frac{1}{\pi_{S',4}}+\frac{2}{\pi_{5,5}}=0\,; \quad . \quad . \quad . \quad (248)$$

or the sum of the reciprocals of the squares of the tangents from the points of intersection of four spheres to the spheres is equal to the reciprocal of the square of the radius of the sphere which cuts them orthogonally.

Again, from equation (228),

$$\frac{\pi_{x,1}}{\pi_{P,1}}+\frac{\pi_{x,2}}{\pi_{Q,2}}+\frac{\pi_{x,3}}{\pi_{R,3}}+\frac{\pi_{x,4}}{\pi_{S,4}}+\frac{\pi_{x,x}}{\pi_{x,4}}=1\,;$$

or if (x) cut $(1, 2, 3, 4)$ at angles ϕ_1, ϕ_2, ϕ_3, ϕ_4, we have

$$\frac{1}{\rho}+\frac{1}{r}\sec\omega=\frac{2r_1\cos\phi_1}{\pi_{P,1}}+\frac{2r_2\cos\phi_2}{\pi_{Q,2}}+\frac{2r_3\cos\phi_3}{\pi_{R,3}}+\frac{2r_4\cos\phi_4}{\pi_{S,4}}. \quad . \quad . \quad . \quad (249)$$

The Spheres which Touch Four given Spheres.—§§ 222, 223.

222. Let the given spheres be denoted by $(1, 2, 3, 4)$, their orthogonal sphere by (5). Then x denoting any sphere which touches them, we shall have by the equation

$$\Pi\begin{pmatrix} x, 1, 2, 3, 4, 5 \\ x, 1, 2, 3, 4, 5 \end{pmatrix}=0,$$

$$\begin{vmatrix} \sin^2\omega, & \sqrt{\pi_{1,1}}, & \sqrt{\pi_{2,2}}, & \sqrt{\pi_{3,3}}, & \sqrt{\pi_{4,4}} \\ \sqrt{\pi_{1,1}}, & \pi_{1,1}, & \pi_{1,2}, & \pi_{1,3}, & \pi_{1,4} \\ \sqrt{\pi_{2,2}}, & \pi_{2,1}, & \pi_{2,2}, & \pi_{2,3}, & \pi_{2,4} \\ \sqrt{\pi_{3,3}}, & \pi_{3,1}, & \pi_{3,2}, & \pi_{3,3}, & \pi_{3,4} \\ \sqrt{\pi_{4,4}}, & \pi_{4,1}, & \pi_{4,2}, & \pi_{4,3}, & \pi_{4,4} \end{vmatrix}=0\,; \quad . \quad . \quad (250)$$

by taking the expression $\sqrt{\pi_{1,1}}$, $\sqrt{\pi_{2,2}}$, &c., with different signs, we obtain the eight values of $\sin^2\omega$ corresponding to the eight pairs of tangent spheres.

The radii are given at once by the formula

$$\Pi\begin{pmatrix} x, 1, 2, 3, 4, 5 \\ \theta, 1, 2, 3, 4, 5 \end{pmatrix}=0.$$

Thus, if ρ be the radius of the sphere touching $(1, 2, 3, 4)$ externally, and ω be the angle at which it cuts the orthogonal sphere, we have

$$\begin{vmatrix} \dfrac{1}{\rho}+\dfrac{\cos\omega}{r}, & \dfrac{1}{r_1}, & \dfrac{1}{r_2}, & \dfrac{1}{r_3}, & \dfrac{1}{r_4} \\[2mm] 1, & -1, & \cos\omega_{1,2}, & \cos\omega_{1,3}, & \cos\omega_{1,4} \\[1mm] 1, & \cos\omega_{2,1}, & -1, & \cos\omega_{2,3}, & \cos\omega_{2,4} \\[1mm] 1, & \cos\omega_{3,1}, & \cos\omega_{3,2}, & -1, & \cos\omega_{3,4} \\[1mm] 1, & \cos\omega_{4,1}, & \cos\omega_{4,2}, & \cos\omega_{4,3}, & -1 \end{vmatrix}=0 \quad . \quad . \quad (251)$$

223. There is no analogous theorem, in general, to FEUERBACH's theorem. In the case, however, when the given spheres (1, 2, 3, 4) cut at angles such that $\cos\omega_{1,2}=\cos\omega_{1,3}=\cos\omega_{1,4}=\alpha$; $\cos\omega_{2,4}=\cos\omega_{4,3}=\cos\omega_{3,2}=\beta$; it may be shown that the two spheres touching the given spheres, all externally, and the eight spheres touching three externally, or three internally, may be divided into two groups of five, each sphere of either group touching a certain other sphere.

Thus, let the tangent spheres be denoted by (a, b, c, d, e), let z denote the sphere which touches them. Suppose z touches (a) internally, and (b, c, d, e) externally.

From the equation

$$\Pi\begin{pmatrix} z, & 1, & 2, 3, 4, 5 \\ x, & a, & b, c, d, e \end{pmatrix}=0,$$

we deduce

$$A\cos\omega_{z,x}+B\cos\omega_{x,1}+C\cos\omega_{x,2}+D\cos\omega_{x,3}+E\cos\omega_{x,4}+F\cos\omega_{x,5}=0 ;$$

where

$$A= \ \ 2\cos\omega_{a,5}-\ \ \cos\omega_{b,5}-\ \ \cos\omega_{c,5}-\ \ \cos\omega_{d,5}-\ \ \cos\omega_{e,5},$$

$$B=-\ \ \cos\omega_{a,5}+2\cos\omega_{b,5}-\ \ \cos\omega_{c,5}-\ \ \cos\omega_{d,5}-\ \ \cos\omega_{e,5},$$

$$C=-\ \ \cos\omega_{a,5}-\ \ \cos\omega_{b,5}+2\cos\omega_{c,5}-\ \ \cos\omega_{d,5}-\ \ \cos\omega_{e,5},$$

$$D=-\ \ \cos\omega_{a,5}-\ \ \cos\omega_{b,5}-\ \ \cos\omega_{c,5}+2\cos\omega_{d,5}-\ \ \cos\omega_{e,5},$$

$$E=-\ \ \cos\omega_{a,5}-\ \ \cos\omega_{b,5}-\ \ \cos\omega_{c,5}-\ \ \cos\omega_{d,5}+2\cos\omega_{e,5},$$

$$F=-6.$$

By taking for x, 1, 2, 3, 4, 5 in succession, we can find $\cos\omega_{z,1}$, $\cos\omega_{z,2}$, $\cos\omega_{z,3}$, $\cos\omega_{z,4}$, $\cos\omega_{z,5}$, and then substituting in the equation obtained by putting z for x, we find the required condition.

By equation (250) we find at once,

$$\cos^2\omega_{a,5}=\frac{3(\alpha+1)^2}{3\alpha^2+2\beta-1} ; \quad \cos^2\omega_{b,5}=\frac{3(\alpha-1)^2}{3\alpha^2+2\beta-1} ;$$

and

$$\cos^2\omega_{c,5}=\cos^2\omega_{d,5}=\cos^2\omega_{e,5}=\frac{(\alpha-1)^2(\beta+1)^2-4(\alpha^2-1)(\beta^2-1)}{(\beta+1)^2(3\alpha^2+2\beta-1)}.$$

Taking $\cos \omega_{c,5}$, $\cos \omega_{d,5}$, $\cos \omega_{e,5}$ with the same sign, we have

$$A = 2 \cos \omega_{a,5} - \cos \omega_{b,5} - 3 \cos \omega_{c,5},$$

$$B = 2 \cos \omega_{b,5} - \cos \omega_{a,5} - 3 \cos \omega_{c,5},$$

$$C = D = E = -\cos \omega_{a,5} - \cos \omega_{b,5}.$$

Hence

$$A \cos \omega_{z,1} = B - 3aC,$$

$$A \cos \omega_{z,2} = A \cos \omega_{z,3} = A \cos \omega_{z,4} = -aB + (1 - 2\beta)C,$$

$$A \cos \omega_{z,5} = -6,$$

and

$$-A + B \cos \omega_{1,z} + 3C \cos \omega_{2,z} + F \cos \omega_{5,z} = 0.$$

Substituting, we must have

$$-A^2 + B^2 - 6aBC + 3C^2(1 - 2\beta) + 36 = 0,$$

or

$$12 + (\cos \omega_{5,b} - \cos \omega_{5,a})(\cos \omega_{5,a} + \cos \omega_{5,b} - 6 \cos \omega_{5,c}) \cdot$$
$$+ 2a(\cos \omega_{5,a} + \cos \omega_{5,b})(2 \cos \omega_{5,b} - \cos \omega_{5,a} - 3 \cos \omega_{5,c})$$
$$- (2\beta - 1)(\cos \omega_{5,b} + \cos \omega_{5,a})^2 = 0.$$

The coefficient of $\cos \omega_{5,c} = (a+1) \cos \omega_{5,b} + (a-1) \cos \omega_{5,a}$, and choosing the signs of $\cos \omega_{5,a}$, $\cos \omega_{5,b}$, so as to make this vanish, we can easily show that this condition is satisfied.

Thus if τ_1, τ'_1 denote the spheres for which $\cos \omega = \pm \dfrac{(a+1)\sqrt{3}}{\sqrt{3a^2 + 2\beta - 1}}$, and if τ_2, τ'_2 denote the spheres for which $\cos \omega = \mp \dfrac{(a-1)\sqrt{3}}{\sqrt{3a^2 + 2\beta - 1}}$, and if (τ_3, τ'_3) (τ_4, τ'_4) (τ_5, τ'_5) denote the other spheres, we see that the groups

$$(\tau_1, \tau_2, \tau_3, \tau_4, \tau_5) \qquad (\tau'_1, \tau'_2, \tau'_3, \tau'_4, \tau'_5)$$
$$(\tau_1, \tau_2, \tau'_3, \tau'_4, \tau'_5) \qquad (\tau'_1, \tau'_2, \tau_3, \tau_4, \tau_5)$$

have each a common tangent sphere, which touches τ_1 or τ'_1 in the opposite sense to τ_2, τ_3, &c.

CHAPTER IV.—POWER-COORDINATES.

Definition.—§§ 224–227.

224. Since a sphere, plane, or point is completely determinate when its powers are known with respect to any five spheres not having a common orthogonal sphere, we may define the coordinates of a point (sphere or plane), referred to such a system of spheres, as any multiples, the same or different, of the powers of the point with respect to them.

Thus if $(xyzwv)$ be the coordinates of any point, whose Cartesian coordinates are $(\alpha\beta\gamma)$, we shall have

$$x \text{ proportional to } (\alpha-a)^2+(\beta-b)^2+(\gamma-c)^2-r^2,$$

where (abc) is the centre of the sphere of reference, r the radius. Thus $(xyzwv)$ are quadric functions of a particular form of the Cartesian coordinates of the centres of the spheres of reference.

We shall find it convenient to restrict the use of (x, y, z, w, v) to denote the coordinates of a point; the coordinates of a sphere will be denoted by $(\xi, \eta, \zeta, \omega, \varpi)$; and the coordinates of a plane by $(\lambda, \mu, \nu, \rho, \sigma)$.

225. Let us denote the system of reference by $(1, 2, 3, 4, 5)$; then if $(xyzwv)$ be the coordinates of any point P, we see that, θ denoting as usual the plane at infinity, so that $\pi_{\theta,\mathrm{P}}=1$, then, since $\pi_{\mathrm{P,P}}=0$, the coordinates of P must satisfy a homogeneous quadric relation

$$\Pi\binom{\mathrm{P}, 1, 2, 3, 4, 5}{\mathrm{P}, 1, 2, 3, 4, 5}=0,$$

and a non-homogeneous linear relation

$$\Pi\binom{\mathrm{P}, 1, 2, 3, 4, 5}{\theta, 1, 2, 3, 4, 5}=0.$$

Let us suppose the coordinates of P defined by the equations

$$x=k_1.\pi_{\mathrm{P},1}, \quad y=k_2.\pi_{\mathrm{P},2}, \quad z=k_3.\pi_{\mathrm{P},3}, \quad w=k_4.\pi_{\mathrm{P},4}, \quad v=k_5.\pi_{\mathrm{P},5}.$$

Then the quadric relation which $(xyzwv)$ must satisfy is

$$\begin{vmatrix} 0 & \dfrac{x}{k_1}, & \dfrac{y}{k_2}, & \dfrac{z}{k_3}, & \dfrac{w}{k_4}, & \dfrac{v}{k_5} \\[2mm] \dfrac{x}{k_1}, & \pi_{1,1}, & \pi_{1,2}, & \pi_{1,3}, & \pi_{1,4}, & \pi_{1,5} \\[2mm] \dfrac{y}{k_2}, & \pi_{2,1}, & \pi_{2,2}, & \pi_{2,3}, & \pi_{2,4}, & \pi_{2,5} \\[2mm] \dfrac{z}{k_3}, & \pi_{3,1}, & \pi_{3,2}, & \pi_{3,3}, & \pi_{3,4}, & \pi_{3,5} \\[2mm] \dfrac{w}{k_4}, & \pi_{4,1}, & \pi_{4,2}, & \pi_{4,3}, & \pi_{4,4}, & \pi_{4,5} \\[2mm] \dfrac{v}{k_5}, & \pi_{5,1}, & \pi_{5,2}, & \pi_{5,3}, & \pi_{5,4}, & \pi_{5,5} \end{vmatrix} = 0 \quad . \quad . \quad . \quad . \quad (252)$$

This is called the equation of the absolute : we will denote it by

$$\psi(x, y, z, w, v) \equiv (a_{1,1}, a_{1,2}, a_{1,3}, a_{1,4} \ldots)(x, y, z, w, v)^2 = 0, \qquad . \quad (253)$$

where

$$a_{1,1} = \frac{1}{k_1{}^2} \frac{-\Pi\begin{pmatrix} 2, 3, 4, 5 \\ 2, 3, 4, 5 \end{pmatrix}}{\Pi\begin{pmatrix} 1, 2, 3, 4, 5 \\ 1, 2, 3, 4, 5 \end{pmatrix}}, \quad a_{1,2} = \frac{1}{k_1 k_2} \frac{-\Pi\begin{pmatrix} 1, 3, 4, 5 \\ 2, 3, 4, 5 \end{pmatrix}}{\Pi\begin{pmatrix} 1, 2, 3, 4, 5 \\ 1, 2, 3, 4, 5 \end{pmatrix}}, \&c. ;$$

and then the linear relation which $(xyzwv)$ must satisfy, may be expressed thus

$$k_1 \frac{\partial \psi}{\partial x} + k_2 \frac{\partial \psi}{\partial y} + k_3 \frac{\partial \psi}{\partial z} + k_4 \frac{\partial \psi}{\partial w} + k_5 \frac{\partial \psi}{\partial v} = -2. \quad . \quad . \quad . \quad . \quad (254)$$

226. If $(\xi, \eta, \zeta, \omega, \varpi)$ be the coordinates of any sphere, S, we see that, since $\pi_{S,\theta} = 1$, the coordinates of S must satisfy a non-homogeneous linear relation

$$\Pi\begin{pmatrix} \theta, 1, 2, 3, 4, 5 \\ S, 1, 2, 3, 4, 5 \end{pmatrix} = 0,$$

which may be written

$$k_1 \frac{\partial \psi}{\partial \xi} + k_2 \frac{\partial \psi}{\partial \eta} + k_3 \frac{\partial \psi}{\partial \zeta} + k_4 \frac{\partial \psi}{\partial \omega} + k_5 \frac{\partial \psi}{\partial \varpi} = -2.. \quad . \quad . \quad . \quad . \quad (255)$$

227. Again, if L denote any plane whose coordinates are $(\lambda, \mu, \nu, \rho, \sigma)$, since $\pi_{L,\theta} = 0$, $\pi_{L,L} = -2$, we see that the coordinates of L must satisfy a linear homogeneous equation and a non-homogeneous quadric relation, viz.,

$$\Pi\begin{pmatrix} L, 1, 2, 3, 4, 5 \\ \theta, 1, 2, 3, 4, 5 \end{pmatrix} = 0,$$

which may be written

$$k_1 \frac{\partial \psi}{\partial \lambda} + k_2 \frac{\partial \psi}{\partial \mu} + k_3 \frac{\partial \psi}{\partial \nu} + k_4 \frac{\partial \psi}{\partial \rho} + k_5 \frac{\partial \psi}{\partial \sigma} = 0 ; \quad . \quad . \quad . \quad . \quad . \quad (256)$$

and

$$\Pi \begin{pmatrix} \text{L}, & 1, & 2, & 3, & 4, & 5 \\ \text{L}, & 1, & 2, & 3, & 4, & 5 \end{pmatrix} = 0,$$

which may be written

$$\lambda \frac{\partial \psi}{\partial \lambda} + \mu \frac{\partial \psi}{\partial \mu} + \nu \frac{\partial \psi}{\partial \nu} + \rho \frac{\partial \psi}{\partial \rho} + \sigma \frac{\partial \psi}{\partial \sigma} = +2,$$

or

$$\psi(\lambda, \mu, \nu, \rho, \sigma) = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (257)$$

The Sphere.—§§ 228–231.

228. Let P be any point on the sphere S whose coordinates are $(\xi \eta \zeta \omega \varpi)$, then since $\pi_{\text{P,S}} = 0$, and

$$\Pi \begin{pmatrix} \text{P}, & 1, & 2, & 3, & 4, & 5 \\ \text{S}, & 1, & 2, & 3, & 4, & 5 \end{pmatrix} = 0,$$

we see that the equation of the sphere is

$$\frac{\partial \psi}{\partial \xi} x + \frac{\partial \psi}{\partial \eta} y + \frac{\partial \psi}{\partial \zeta} z + \frac{\partial \psi}{\partial \omega} w + \frac{\partial \psi}{\partial \varpi} v = 0 ;$$

and hence the general equation of the first degree,

$$ax + by + cz + dw + ev = 0,$$

will, in general, represent a sphere, whose coordinates are given by

$$\frac{\frac{\partial \psi}{\partial \xi}}{a} = \frac{\frac{\partial \psi}{\partial \eta}}{b} = \frac{\frac{\partial \psi}{\partial \zeta}}{c} = \frac{\frac{\partial \psi}{\partial \omega}}{d} = \frac{\frac{\partial \psi}{\partial \varpi}}{e} = \frac{-2}{ak_1 + bk_2 + ck_3 + dk_4}, \quad . \quad . \quad . \quad . \quad (258)$$

by equation (255).

229. Given any two spheres $(\xi \eta \zeta \omega \varpi)$ $(\xi' \eta' \zeta' \omega' \varpi')$, their power π is, since

$$\Pi \begin{pmatrix} \text{S}, & 1, & 2, & 3, & 4, & 5 \\ \text{S}, & 1, & 2, & 3, & 4, & 5 \end{pmatrix} = 0,$$

given by

$$-2\pi = \xi' \frac{\partial \psi}{\partial \xi} + \eta' \frac{\partial \psi}{\partial \eta} + \zeta' \frac{\partial \psi}{\partial \zeta} + \omega' \frac{\partial \psi}{\partial \omega} + \varpi' \frac{\partial \psi}{\partial \varpi}. \quad . \quad . \quad . \quad . \quad (259)$$

In particular the radius of the sphere $(\xi\eta\zeta\omega\varpi)$ will be given by

$$2r^2 = \psi(\xi,\ \eta,\ \zeta,\ \omega,\ \varpi). \qquad \ldots \ldots \ldots \quad (260)$$

230. Hence the radius of the sphere

$$ax + by + cz + dw + ev = 0,$$

is given by

$$\begin{vmatrix} a_{1,1}, & a_{1,2}, & a_{1,3}, & a_{1,4}, & a_{1,5}, & a \\ a_{2,1}, & a_{2,2}, & a_{2,3}, & a_{2,4}, & a_{2,5}, & b \\ a_{3,1}, & a_{3,2}, & a_{3,3}, & a_{3,4}, & a_{3,5}, & c \\ a_{4,1}, & a_{4,2}, & a_{4,3}, & a_{4,4}, & a_{4,5}, & d \\ a_{5,1}, & a_{5,2}, & a_{5,3}, & a_{5,4}, & a_{5,5}, & e \\ a, & b, & c, & d, & e, & u \end{vmatrix} = 0, \quad \ldots \ldots \quad (261)$$

where

$$u = 2r^2(ak_1 + bk_2 + ck_3 + dk_4 + ek_5)^2.$$

We shall find it convenient to denote the bordered Hessian of ψ by Ψ; and suppose the coefficients so determined that we may express equation (261) thus—

$$u = \Psi(a,\ b,\ c,\ d,\ e) \qquad \ldots \ldots \ldots \quad (262)$$

231. The power of the sphere $(\xi\eta\zeta\omega\varpi)$ with respect to the sphere

$$ax + by + cz + dw + ev = 0,$$

is clearly given by

$$\pi = \frac{a\xi + b\eta + c\zeta + d\omega + e\varpi}{ak_1 + bk_2 + ck_3 + dk_4 + ek_5}; \qquad \ldots \ldots \ldots \quad (263)$$

and the power of the two spheres

$$ax + by + cz + dw + ev = 0,$$
$$a'x + b'y + c'z + d'w + e'v = 0,$$

is clearly given by

$$\pi = \frac{a'\dfrac{\partial\Psi}{\partial a} + b'\dfrac{\partial\Psi}{\partial b} + c'\dfrac{\partial\Psi}{\partial c} + d'\dfrac{\partial\Psi}{\partial d} + e'\dfrac{\partial\Psi}{\partial e}}{(ak_1 + bk_2 + ck_3 + dk_4 + ek_5)(a'k_1 + b'k_2 + c'k_3 + d'k_4 + e'k_5)}.$$

And so the angle of intersection of the spheres will be given by

$$\cos\phi = -\frac{1}{2}\frac{a'\dfrac{\partial\Psi}{\partial a} + b'\dfrac{\partial\Psi}{\partial b} + c'\dfrac{\partial\Psi}{\partial c} + d'\dfrac{\partial\Psi}{\partial d} + e'\dfrac{\partial\Psi}{\partial e}}{\sqrt{\Psi(a,\ b,\ c,\ d,\ e).\Psi(a',\ b',\ c',\ d',\ e')}}. \qquad \ldots \ldots \quad (264)$$

The Plane.—§§ 232–235.

232. If $(\lambda\mu\nu\rho\sigma)$ be any plane L, $(xyzwv)$ any point P on it, we have at once, since

$$\Pi\begin{pmatrix} \text{P, 1, 2, 3, 4, 5} \\ \text{L, 1, 2, 3, 4, 5} \end{pmatrix}=0,$$

$$x\frac{\partial\psi}{\partial\lambda}+y\frac{\partial\psi}{\partial\mu}+z\frac{\partial\psi}{\partial\nu}+w\frac{\partial\psi}{\partial\rho}+v\frac{\partial\psi}{\partial\sigma}=0.$$

But by (256)

$$k_1\frac{\partial\psi}{\partial\lambda}+k_2\frac{\partial\psi}{\partial\mu}+k_3\frac{\partial\psi}{\partial\nu}+k_4\frac{\partial\psi}{\partial\rho}+k_5\frac{\partial\psi}{\partial\sigma}=0 ;$$

hence the equation

$$ax+by+cz+dw+ev=0,$$

will represent a plane when

$$ak_1+bk_2+ck_3+dk_4+ek_5=0. \quad\quad\quad\quad (265)$$

And if this condition be satisfied, we have to determine its coordinates,

$$\frac{\frac{\partial\psi}{\partial\lambda}}{a}=\frac{\frac{\partial\psi}{\partial\mu}}{b}=\frac{\frac{\partial\psi}{\partial\nu}}{c}=\frac{\frac{\partial\psi}{\partial\rho}}{d}=\frac{\frac{\partial\psi}{\partial\sigma}}{e}=\frac{2}{\sqrt{\Psi(a,\,b,\,c,\,d,\,e)}}, \quad\quad (266)$$

where $\Psi(a,\,b,\,c,\,d,\,e)$ is defined by equation (262).

233. The power of the sphere $(\xi\eta\zeta\omega\varpi)$ and the plane

$$ax+by+cz+dw+ev=0$$

is given by

$$-2\pi=\xi\frac{\partial\psi}{\partial\lambda}+\eta\frac{\partial\psi}{\partial\mu}+\zeta\frac{\partial\psi}{\partial\nu}+\omega\frac{\partial\psi}{\partial\rho}+\varpi\frac{\partial\psi}{\partial\sigma} ;$$

therefore

$$\pi=\frac{a\xi+b\eta+c\zeta+d\omega+e\varpi}{\sqrt{\Psi(a,\,b,\,c,\,d,\,e)}}. \quad\quad\quad\quad (267)$$

234. The angle of intersection of the planes

$$ax+by+cz+dw+ev=0,$$
$$a'x+b'y+c'z+d'w+e'v=0,$$

will be given by

$$\cos\phi=-\tfrac{1}{2}\frac{a'\frac{\partial\Psi}{\partial a}+b'\frac{\partial\Psi}{\partial b}+c'\frac{\partial\Psi}{\partial c}+d'\frac{\partial\Psi}{\partial d}+e'\frac{\partial\Psi}{\partial e}}{\sqrt{\Psi(a,\,b,\,c,\,d,\,e).\Psi(a',\,b',\,c',\,d',\,e')}}. \quad\quad\quad (268)$$

235. The coordinates of the plane at infinity are clearly k_1, k_2, k_3, k_4, k_5, and so the equation of the plane at infinity is

$$x\frac{\partial\psi}{\partial k_1}+y\frac{\partial\psi}{\partial k_2}+z\frac{\partial\psi}{\partial k_3}+w\frac{\partial\psi}{\partial k_4}+v\frac{\partial\psi}{\partial k_5}=0.$$

The Point.—§§ 236–238.

236. The power of the point $(x\,y\,z\,w\,v)$ with respect to the surface

$$ax+by+cz+dw+ev=0,$$

will be equal to

$$\frac{ax+by+cz+dw+ev}{ak_1+bk_2+ck_3+dk_4+ek_5},$$

if the surface is a sphere, and will be equal to

$$\frac{ax+by+cz+dw+ev}{\sqrt{\Psi(a,\,b,\,c,\,d,\,e)}},$$

if the surface is a plane.

237. The power of the two points $(x\,y\,z\,w\,v)$, $(x'y'z'w'v')$ will be given by

$$-2\pi=x'\frac{\partial\psi}{\partial x}+y'\frac{\partial\psi}{\partial y}+z'\frac{\partial\psi}{\partial z}+w'\frac{\partial\psi}{\partial w}+v'\frac{\partial\psi}{\partial v}.$$

Hence, since

$$\psi(x,\,y,\,z,\,w,\,v)=\psi(x',\,y',\,z',\,w',\,v')=0,$$

the distance δ between the two points will be given by

$$2\delta^2=\psi\{x-x',\,y-y',\,z-z',\,w-w',\,v-v'\}.\quad\ldots\quad\ldots\quad(269)$$

238. Let P, Q, R, S be any four points, then by equation (216) we have for the volume of the tetrahedron formed by them,

$$288.\{V(P,\,Q,\,R,\,S)\}^2=\Pi\binom{\theta,\,P,\,Q,\,R,\,S}{\theta,\,P,\,Q,\,R,\,S};$$

and by § 204,

$$\Pi\binom{\theta,\,P,\,Q,\,R,\,S}{\theta,\,P,\,Q,\,R,\,S}\times\Pi\binom{1,\,2,\,3.\,4,\,5}{1,\,2,\,3,\,4,\,5}=\left\{\Pi\binom{\theta,\,P,\,Q,\,R,\,S}{1,\,2,\,3,\,4,\,5}\right\}^2.$$

If then the coordinates of P, Q, R, S referred to (1, 2, 3, 4, 5) be $(x_1y_1z_1w_1v_1)$ $(x_2y_2z_2w_2v_2)(x_3y_3z_3w_3v_3)(x_4y_4z_4w_4v_4)$, we see that we shall have

$$V(P, Q, R, S) = \mu \begin{vmatrix} x_1, & y_1, & z_1, & w_1, & v_1 \\ x_2, & y_2, & z_2, & w_2, & v_2 \\ x_3, & y_3, & z_3, & w_3, & v_3 \\ x_4, & y_4, & z_4, & w_4, & v_4 \\ k_1, & k_2, & k_3, & k_4, & k_5 \end{vmatrix} ; \qquad \ldots \ldots (270)$$

where $288\mu^2 k_1^2 k_2^2 k_3^2 k_4^2 k_5^2 . \Pi \begin{pmatrix} 1, 2, 3, 4, 5 \\ 1, 2, 3, 4, 5 \end{pmatrix} = 1.$

Coordinate Systems of Reference.—§§ 239, 240.

239. The most convenient system of reference is five spheres which are mutually orthotomic : this may be called the orthogonal system. If r_1, r_2, r_3, r_4, r_5 are the radii of five such spheres, it is simplest to take k_1, k_2, k_3, k_4, k_5, the coordinates of the plane at infinity, as inversely proportional to them. In this case we shall have, by § 212,

$$\left. \begin{aligned} 2.\psi(x, y, z, w, v) &\equiv x^2 + y^2 + z^2 + w^2 + v^2 \\ 2\,\Psi(x, y, z, w, v) &\equiv x^2 + y^2 + z^2 + w^2 + v^2 \end{aligned} \right\} \qquad \ldots \ldots (271)$$

Thus the angle between the spheres

$$ax + by + cz + dw + ev = 0,$$
$$a'x + b'y + c'z + d'w + e'v = 0,$$

will be given by

$$\cos \phi = -\frac{aa' + bb' + cc' + dd' + ee'}{\sqrt{(a^2 + b^2 + c^2 + d^2 + e^2)(a'^2 + b'^2 + c'^2 + d'^2 + e'^2)}}.$$

240. It is, however, very often convenient to take as the system of reference three spheres cutting orthogonally, and their two points of intersection. And then if r_1, r_2, r_3 be the radii of the spheres, ϵ the distance between their points of intersection, it is simplest to take k_1, k_2, k_3, k_4, k_5, the coordinates of the plane at infinity, as inversely proportional to r_1, r_2, r_3, ϵ, ϵ : so that we shall have by § 214,

$$\left. \begin{aligned} 2\psi(x, y, z, w, v) &\equiv x^2 + y^2 + z^2 - 4vw \\ 2\Psi(x, y, z, w, v) &\equiv x^2 + y^2 + z^2 - vw \end{aligned} \right\} ; \qquad \ldots \ldots (272)$$

so that the angle between the spheres

$$ax + by + cz + dw + ev = 0,$$
$$a'x + b'y + c'z + d'w + e'v = 0,$$

will be given by

$$\cos \phi = -\frac{aa' + bb' + cc' - \frac{1}{2}(ed' + e'd)}{\sqrt{(a^2 + b^2 + c^2 - de)(a'^2 + b'^2 + c'^2 - d'e')}}.$$

CHAPTER V.—GENERAL EQUATION OF THE SECOND DEGREE IN POWER-COORDINATES.

Nature of the Surface.—§§ 241, 242.

241. The most general equation of the second degree in power-coordinates may be written

$$\phi(x, y, z, w, v) \equiv \alpha_{1,1}\, x^2 + \alpha_{2,2}\, y^2 + \alpha_{3,3}\, z^2 + \alpha_{4,4}\, w^2 + \alpha_{5,5}\, v^2$$
$$+ 2\alpha_{1,2}\, xy + 2\alpha_{1,3}\, xz + 2\alpha_{1,4}\, xw + 2\alpha_{1,5}\, xv$$
$$+ 2\alpha_{2,3}\, yz + 2\alpha_{2,4}\, yw + 2\alpha_{2,5}\, yv$$
$$+ 2\alpha_{3,4}\, zw + 2\alpha_{3,5}\, zv + 2\alpha_{4,5}\, vw = 0 ;$$

and since the coordinates of any point must satisfy the equation of the absolute, which is also of the second degree, we see that this equation contains only 14 constants.

Now (X, Y, Z) denoting Cartesian coordinates of a point, we can express the power-coordinates $(xyzwv)$ of the point as linear functions of $X^2 + Y^2 + Z^2$, X, Y, Z, 1 ; and if we substitute for $(xyzwv)$, the equation becomes of the form

$$(X^2 + Y^2 + Z^2)^2 + U_1(X^2 + Y^2 + Z^2) + U_2 = 0,$$

where U_1 is of the first degree, U_2 of the second degree in (X, Y, Z) ; this equation has 14 constants, and represents a surface having the circle at infinity as a nodal curve, and is usually called a cyclide. It thus appears that $\phi = 0$, is a form to which the equation of every cyclide can be reduced.

242. It is also evident that, since the equation of every plane is satisfied by the coordinates of the plane at infinity, the surface $\phi = 0$ will represent a cubic surface and the plane at infinity, if the equation $\phi = 0$ be satisfied by the coordinates of the plane at infinity. Such surfaces have been called cubic cyclides ; they intersect the plane at infinity in the imaginary circle, and also a straight line.

Equation of Tangent at any Point.—§§ 243–247.

243. If $(\xi\, \eta\, \zeta\, \omega\, \varpi)$ be the coordinates of any sphere touching the cyclide ϕ at the point $(x'y'z'w'v')$, we must have

$$\frac{\dfrac{\partial\psi}{\partial\xi}}{\dfrac{\partial\phi}{\partial x'} + k\dfrac{\partial\psi}{\partial x'}} = \frac{\dfrac{\partial\psi}{\partial\eta}}{\dfrac{\partial\phi}{\partial y'} + k\dfrac{\partial\psi}{\partial y'}} = \frac{\dfrac{\partial\psi}{\partial\zeta}}{\dfrac{\partial\phi}{\partial z'} + k\dfrac{\partial\psi}{\partial z'}} = \frac{\dfrac{\partial\psi}{\partial\omega}}{\dfrac{\partial\phi}{\partial w'} + k\dfrac{\partial\psi}{\partial w'}} = \frac{\dfrac{\partial\psi}{\partial\varpi}}{\dfrac{\partial\phi}{\partial v'} + k\dfrac{\partial\psi}{\partial v'}}. \quad . \quad . \quad . \quad (273)$$

And thus the equation of any tangent sphere is of the form,

$$\left(\frac{\partial\phi}{\partial x'}+k\frac{\partial\psi}{\partial x'}\right)x+\left(\frac{\partial\phi}{\partial y'}+k\frac{\partial\psi}{\partial y'}\right)y+\left(\frac{\partial\phi}{\partial z'}+k\frac{\partial\psi}{\partial z'}\right)z+\left(\frac{\partial\phi}{\partial w'}+k\frac{\partial\psi}{\partial w'}\right)w+\left(\frac{\partial\phi}{\partial v'}+k\frac{\partial\psi}{\partial v'}\right)v=0.$$

Hence the equation of the tangent plane at the point $(x'y'z'w'v')$ is

$$\left(k_1\frac{\partial\phi}{\partial x'}+k_2\frac{\partial\phi}{\partial y'}+k_3\frac{\partial\phi}{\partial z'}+k_4\frac{\partial\phi}{\partial w'}+k_5\frac{\partial\phi}{\partial v'}\right)\left(x\frac{\partial\psi}{\partial x'}+y\frac{\partial\psi}{\partial y'}+z\frac{\partial\psi}{\partial z'}+w\frac{\partial\psi}{\partial w'}+v\frac{\partial\psi}{\partial v'}\right)$$

$$=\left(k_1\frac{\partial\psi}{\partial x'}+k_2\frac{\partial\psi}{\partial y'}+k_3\frac{\partial\psi}{\partial z'}+k_4\frac{\partial\psi}{\partial w'}+k_5\frac{\partial\psi}{\partial v'}\right)\left(x\frac{\partial\phi}{\partial x'}+y\frac{\partial\phi}{\partial y'}+z\frac{\partial\phi}{\partial z'}+w\frac{\partial\phi}{\partial w'}+v\frac{\partial\phi}{\partial v'}\right). \quad . \quad . \quad (274)$$

244. The sphere given by the equation

$$\left(x\frac{\partial}{\partial x'}+y\frac{\partial}{\partial y'}+z\frac{\partial}{\partial z'}+w\frac{\partial}{\partial w'}+v\frac{\partial}{\partial v'}\right)(\phi+k\psi)=0,$$

will clearly touch the cyclide at the point $(x''y''z''w''v'')$ if

$$\frac{\frac{\partial\phi}{\partial x'}+k\frac{\partial\psi}{\partial x'}}{\frac{\partial\phi}{\partial x''}+k\frac{\partial\psi}{\partial x''}}=\frac{\frac{\partial\phi}{\partial y'}+k\frac{\partial\psi}{\partial y'}}{\frac{\partial\phi}{\partial y''}+k\frac{\partial\psi}{\partial y''}}=\frac{\frac{\partial\phi}{\partial z'}+k\frac{\partial\psi}{\partial z'}}{\frac{\partial\phi}{\partial z''}+k\frac{\partial\psi}{\partial z''}}=\frac{\frac{\partial\phi}{\partial w'}+k\frac{\partial\psi}{\partial w'}}{\frac{\partial\phi}{\partial w''}+k\frac{\partial\psi}{\partial w''}}=\frac{\frac{\partial\phi}{\partial v'}+k\frac{\partial\psi}{\partial v'}}{\frac{\partial\phi}{\partial v''}+k\frac{\partial\psi}{\partial v''}};$$

in which case k must satisfy the quintic

$$\mathrm{H}(\phi+k\psi)=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (275)$$

where $\mathrm{H}(u)$ denotes the Hessian of u.

We infer, then, that there are in general five systems of bitangent spheres of a cyclide; i.e., of the whole number of tangent spheres at any point of the surface five touch the surface elsewhere. Moreover it is evident that the system of bitangent spheres corresponding to a particular root of (275) all cut a certain fixed sphere orthogonally, the coordinates of this sphere being proportional to the minors of the constituents of any row in the determinant $\mathrm{H}(\phi+k\psi)$.

245. If the coordinates of a bitangent sphere satisfy the absolute, i.e., if the radius of the sphere be indefinitely small, the sphere may be considered as a focus. It is evident that the foci belonging to any system of bitangents will form a curve of the second degree on the corresponding orthogonal sphere. Such a curve will be a spheri-quadric.

246. If the coordinates of a bitangent sphere satisfy the condition, for a plane, the corresponding equation will represent a double tangent plane; and its coordinates

must satisfy an equation of the second degree, and thus we shall have a double tangent cone corresponding to each system of bitangent spheres, the vertex of the cone being the centre of the sphere which cuts the particular system of bitangents orthogonally.

247. Again, it is clear that if by any transformation the equation of the surface becomes Φ, and the equation of the absolute Ψ, then the same value of k which satisfies $H(\phi+k\psi)=0$ must also satisfy $H(\Phi+k\Psi)=0$; and hence the coefficients of the powers of k in the equation $H(\phi+k\psi)=0$ are invariants.

Equation of Normal at any Point.—§§ 248–250.

248. Let $(\xi \eta \zeta \omega \varpi)$ be the coordinates of any sphere which cuts the surface $\phi(x\,y\,z\,w\,v)=0$ orthogonally at the point $(x'y'z'w'v')$, we must have

$$\left(\xi\frac{\partial}{\partial x'}+\eta\frac{\partial}{\partial y'}+\zeta\frac{\partial}{\partial z'}+\omega\frac{\partial}{\partial w'}+\varpi\frac{\partial}{\partial v'}\right)(\phi+k\psi)=0, \quad . \quad . \quad . \quad . \quad (276)$$

for all values of k.

It follows that if $(\lambda\,\mu\,\nu\,\rho\,\sigma)$ be any plane containing the normal to the cyclide at the point $(x'y'z'w'v')$, we must have

$$\lambda\frac{\partial\phi}{\partial x'}+\mu\frac{\partial\phi}{\partial y'}+\nu\frac{\partial\phi}{\partial z'}+\rho\frac{\partial\phi}{\partial w'}+\sigma\frac{\partial\phi}{\partial v'}=0,$$

$$\lambda\frac{\partial\psi}{\partial x'}+\mu\frac{\partial\psi}{\partial y'}+\nu\frac{\partial\psi}{\partial z'}+\rho\frac{\partial\psi}{\partial w'}+\sigma\frac{\partial\psi}{\partial v'}=0,$$

$$\lambda\frac{\partial\psi}{\partial k_1}+\mu\frac{\partial\psi}{\partial k_2}+\nu\frac{\partial\psi}{\partial k_3}+\rho\frac{\partial\psi}{\partial k_4}+\sigma\frac{\partial\psi}{\partial k_5}=0.$$

249. If we take as our system of reference an orthogonal system of spheres, radii $(r_1, r_2, r_3, r_4, r_5)$, then, by forming equations to planes containing the normal at $(x'y'z'w'v')$ and passing through the centres of the spheres of reference, we easily obtain for the equations of the normal,

$$\begin{Vmatrix} x, & y, & z, & w, & v \\ \dfrac{\partial\phi}{\partial x''} & \dfrac{\partial\phi}{\partial y''} & \dfrac{\partial\phi}{\partial z''} & \dfrac{\partial\phi}{\partial w''} & \dfrac{\partial\phi}{\partial v'} \\ x', & y', & z', & w', & v' \\ \dfrac{1}{r_1}, & \dfrac{1}{r_2}, & \dfrac{1}{r_3}, & \dfrac{1}{r_4}, & \dfrac{1}{r_5} \end{Vmatrix} =0 \quad . \quad . \quad . \quad . \quad . \quad (277)$$

250. Again, from (277), we see that if a sphere cut the cyclide normally at the point $(x'y'z'w'v')$, we must have

$$x'\frac{\partial\phi}{\partial\xi}+y'\frac{\partial\phi}{\partial\eta}+z'\frac{\partial\phi}{\partial\zeta}+w'\frac{\partial\phi}{\partial\omega}+v'\frac{\partial\phi}{\partial\varpi}=0,$$

$$x'\frac{\partial\psi}{\partial\xi}+y'\frac{\partial\psi}{\partial\eta}+z'\frac{\partial\psi}{\partial\zeta}+w'\frac{\partial\psi}{\partial\omega}+v'\frac{\partial\psi}{\partial\varpi}=0.$$

If then $(\xi\,\eta\,\zeta\,\omega\,\varpi)$ be chosen, so that

$$\frac{\dfrac{\partial \phi}{\partial \xi}}{\dfrac{\partial \psi}{\partial \xi}} = \frac{\dfrac{\partial \phi}{\partial \eta}}{\dfrac{\partial \psi}{\partial \eta}} = \frac{\dfrac{\partial \phi}{\partial \zeta}}{\dfrac{\partial \psi}{\partial \zeta}} = \frac{\dfrac{\partial \phi}{\partial \omega}}{\dfrac{\partial \psi}{\partial \omega}} = \frac{\dfrac{\partial \phi}{\partial \varpi}}{\dfrac{\partial \psi}{\partial \varpi}} = -\mu \text{ say,}$$

then the sphere $(\xi \eta \zeta \omega \varpi)$ will cut the cyclide orthogonally at every point of its curve of intersection.

We see at once that μ must satisfy the equation

$$H(\phi + \mu \psi) = 0,$$

and thus there are five such spheres; and the coordinates of the sphere which corresponds to a particular value of μ are proportional to the minors of the constituents of any row in the determinant $H(\phi + \mu \psi)$.

These spheres are evidently the same as those which were mentioned in § 244, as being orthogonal to the five systems of bitangent spheres. We can easily prove by a similar process to that in § 81, that any two of these spheres cut orthogonally. They may be called the principal spheres of the surface,

The Principal Spheres.—§§ 251, 252.

251. By § 244, we see that at every point on the surface of a cyclide can be drawn a tangent sphere cutting one of the principal spheres orthogonally, and touching the surface elsewhere; and hence it follows that the surface must be its own inverse with respect to each principal sphere. Hence these species of surfaces have been called by MOUTARD anallagmatic surfaces, and the principal spheres have been called by CASEY the spheres of inversion.

We have seen that the principal spheres cut the cyclide orthogonally, and it is evident that at points along the curve of section the corresponding bitangent sphere will not touch the cyclide elsewhere, but the curve of section will be a line of curvature on the cyclide.

252. If a cyclide have a node, then, by the principle of inversion, this node must lie on each principal sphere; and thus in this case there can be but three principal spheres, and the node will be one of their points of intersection.

If a cyclide have two nodes, they must be the two points of intersection of the three principal spheres, and any other two spheres forming with these an orthogonal system may be regarded as principal spheres; this case corresponding to that of a quadric of revolution. Similarly if the cyclide have four nodes they occur in pairs, and lie on the only principal sphere; but if we denote the nodes by P, P′, Q, Q′; and the principal sphere by S; then any pair of spheres orthogonal to S and passing through P, P′, which with any pair orthogonal to S and passing through Q, Q′, make up an orthogonal system, may be considered as principal spheres.

But if a cyclide have three nodes, then there are only three principal spheres.

There are also cyclides with only two, and with only one principal sphere.

Reduction of the General Equation to its Simplest Form.—§§ 253–255.

253. We have seen that the number of principal spheres is the same as the number of roots of the discriminating quintic $H(\phi + k\psi) = 0$. Thus in general there are five principal spheres, and we have seen that these cut orthogonally, and it is clear that if we take these spheres as our system of reference, we can express the equation of the cyclide in the form*

$$ax^2 + by^2 + cz^2 + dw^2 + ev^2 = 0;$$

and we might expect, perhaps, that this equation would still be the simplest form, when two or more roots of the equation $H(\phi + k\psi) = 0$ are equal. But five orthogonal spheres cannot be all real, one must be imaginary; and we shall find that if one of the principal spheres corresponding to the unequal roots is imaginary, then this form is the simplest form of the equation; but if all the principal spheres corresponding to the unequal roots are real it is not the simplest form.

254. Let us suppose that two of the roots of $H = 0$ are equal; then, taking for our system of reference the three principal spheres (x, y, z), say corresponding to the unequal roots, and any two other spheres (w, v) forming with them an orthogonal system, we can at once reduce the equation of the surface to the form

$$ax^2 + by^2 + cz^2 + dw^2 + ev^2 + 2nvw = 0;$$

and the discriminating quintic is

$$H \equiv (a+k)(b+k)(c+k)\{(d+k)(e+k) - n^2\} = 0,$$

which can only have equal roots provided $d = e$, $n = 0$; *i.e.*, supposing n is real. Thus, supposing one of the three spheres (x, y, z) to be imaginary, so that (w, v) must be real, the equation can be reduced to the form

$$ax^2 + by^2 + cz^2 + dw^2 + dv^2 = 0,$$

or since the equation of the absolute is

$$x^2 + y^2 + z^2 + w^2 + v^2 = 0,$$

the equation of the cyclide can be put in the form

$$ax^2 + by^2 + cz^2 = 0;$$

and we see that each of the points common to (x, y, z) is a node; thus the surface has two nodes; and, moreover, any two spheres which with (x, y, z) make up an orthogonal system may be taken as principal spheres.

Similarly, if the sphere (x) be imaginary, and the discriminating quintic $H = 0$ has two pairs of equal roots, the equation can be reduced to the form

$$ax^2 + b(y^2 + z^2) + d(w^2 + v^2) = 0;$$

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* [The equation of a cyclide was first given in this form by CASEY (1871) (' Phil. Trans.,' vol. 161, p. 600).—October, 1886.]

and we see that each of the points $x=y=z=0$, and $x=y=w=0$, are nodes; this surface is therefore quadrinodal.

Let us now suppose that each of the spheres (x, y, z) is real, so that the coefficient n must be imaginary. It is simplest to take for system of reference the three principal spheres (x, y, z) and their two points of intersection (w, v), so that the equation of the absolute is

$$x^2+y^2+z^2=4wv.$$

The equation $H(\phi+k\psi)=0$ becomes now

$$(a+k)(b+k)(c+k)\{de-(2k-n)^2\}=0,$$

which has equal roots if either d or $e=0$, so that the equation may be reduced to the form

$$ax^2+by^2+cz^2+dw^2+2nwv=0;$$

or by means of the absolute to the form

$$ax^2+by^2+cz^2+dw^2=0.$$

The surface represented by this equation has only three principal spheres and it has one node, viz., $w=0$.

Similarly, if the discriminating quintic have two pairs of equal roots, and the sphere x correspond to the unequal root, we can show that x being real, the equation can be reduced to the form

$$ax^2+by^2+bz^2+dw^2=0;$$

which represents a surface having three nodes, viz., the point (w) and the points common to the spheres (x, u, u'); u, u' being such that they form with (x, y, z) an orthogonal system.

255. Let us now suppose that three of the roots of the quintic $H(\phi+k\psi)=0$ are equal. Taking for spheres of reference (x, y) corresponding to the remaining roots, and (z, w, v) any other spheres forming with (x, y) an orthogonal system, then the equation of the surface must take the form

$$ax^2+by^2+cz^2+dw^2+ev^2+2fwv+2gzv+2hzw=0,$$

and the equation of the absolute the form

$$x^2+y^2+z^2+w^2+v^2=0;$$

so that the discriminating quintic becomes

$$(k+a)(k+b) \begin{vmatrix} k+c & h & g \\ h & k+d & f \\ g & f & k+e \end{vmatrix} =0.$$

This can only have equal roots if the coefficients h, g, f are real, and then we must have

$$h=f=g=0, \quad c=d=e.$$

In this case one of the two spheres (x, y) must be imaginary, and so the equation reduces to

$$ax^2+by^2+c(z^2+w^2+v^2)=0,$$

which clearly represents two spheres.

Similarly, if the quintic $H=0$ has four equal roots, and x be imaginary, the equation will reduce to

$$ax^2+by^2+bz^2+bw^2+bv^2=0,$$

which represents the imaginary sphere $x=0$.

Let us now suppose the spheres (x, y) to be real, then let us take z any sphere cutting them orthogonally, and let (w, v) be the two points of intersection of (x, y, z): so that the equation of the absolute is

$$x^2+y^2+z^2=4wv,$$

and the discriminating quintic H becomes

$$(a+k)(b+k) \begin{vmatrix} k+c, & h, & g \\ h, & d, & f-2k \\ g, & f-2h, & e \end{vmatrix} =0 ;$$

which can only have equal roots provided that

$$e=g=0, \quad f=-2c ;$$

and then the equation takes the form

$$ax^2+by^2+cz^2+dw^2+2hzw-4cwv=0.$$

By taking, instead of $z=0$, the sphere $z-\dfrac{d}{2h}w=0$, this equation may clearly be reduced to the form

$$ax^2+by^2+2hzw=0.$$

The surface represented by this equation has the point $w=0$ for a *binode*: if a or b is zero, the node is a *unode*.

Again, if the equation $H(\phi+k\psi)=0$ has four equal roots and the sphere corresponding to the remaining root is real: let us take this sphere as $x=0$, and let us take for our system of reference any two spheres (y, z) and the two points (w, v) in which the spheres (x, y, z) intersect, so that the system is semi-orthogonal: then the equation of the surface must be of the form

$$ax^2+by^2+cz^2+dw^2+ev^2+2fzw+2gwy+2hyz+2lyv+2mzv+2nwv=0 ;$$

and since the absolute is of the form

$$x^2+y^2+z^2=4wv,$$

the discriminating quintic becomes

$$(k+a) \begin{vmatrix} k+b, & h, & g, & l \\ h, & k+c, & f, & m \\ g, & f, & d, & n-2k \\ l, & m, & n-2k, & e \end{vmatrix} = 0.$$

This can only have four equal roots if

$$e=l=m=0, \quad 2b=2c=-n.$$

Thus the equation can be reduced to the form

$$ax^2+by^2+bz^2+dw^2+2fwz+2gwy+2hyz-4bwv=0.$$

This may also be written

$$ax^2+dw^2+2fwz+2gwy+2hyz=0 \ ;$$

and by suitably choosing for spheres of reference $(y+\lambda w, z+\mu w)$ instead of (y, z) this equation can evidently be reduced to the form

$$ax^2+2hyz+dw^2=0.$$

The point $w=0$ is a node on the surface: it is in general a *cnic*-node; but if $a=0$ it is a *binode*, and if $h=0$ it is a *unode*.

Thus there are four distinct forms of cyclides: their equations can be reduced to one of the four forms

(A.) $$ax^2+by^2+cz^2+dw^2+ev^2=0;$$

the absolute being

$$x^2+y^2+z^2+w^2+v^2=0.$$

There are five principal spheres: if $d=e$ there are two nodes, and if $b=c, d=e,$ there are four nodes.

(B.) $$ax^2+by^2+cz^2+dw^2=0,$$

the absolute being

$$x^2+y^2+z^2=4wv.$$

This is the general inverse of a quadric surface: there are three principal spheres and one *cnic*-node. If $b=c$ there are three *cnic*-nodes, this is the inverse of a quadric of revolution.

(C.)
$$ax^2 + by^2 + 2hwz = 0,$$

the absolute being

$$x^2 + y^2 + z^2 = 4wv.$$

This surface has two principal spheres and a *binode* ($w = 0$).   If $a = b$ there will be two other nodes.

(D.)
$$ax^2 + 2hyz + dw^2 = 0,$$

the absolute being

$$x^2 + y^2 + z^2 = 4wv.$$

This surface has only one principal sphere and a *cnic*-node.   Also, since the general equation represents a cubic cyclide, when it is satisfied by the coordinates of the plane at infinity, it follows that the equation of a cubic cyclide can always be reduced to one of the four forms given above.

CHAPTER VI.—CLASSIFICATION OF CYCLIDES.

Cyclides have been classified by DARBOUX and CASEY according to the nature of the focal quadrics.   We shall find it more convenient here to discuss the different forms of cyclides in order, according to the number of different roots which the discriminating quintic has.   We have seen that there are four distinct forms to which the equation of a cyclide may be reduced; and it is proposed to discuss briefly the different species represented by similar equations, and a few of their properties.

A. *The General Cyclide.*—§§ 256–264.

256. The equation of the surface is of the form

$$ax^2 + by^2 + cz^2 + dw^2 + ev^2 = 0,$$

the equation of the absolute being

$$x^2 + y^2 + z^2 + w^2 + v^2 = 0 \ ;$$

and the coordinates of the plane at infinity $\dfrac{1}{r_1}, \dfrac{1}{r_2}, \dfrac{1}{r_3}, \dfrac{1}{r_4}, \dfrac{1}{r_5}$, *i.e.*, the reciprocals of the radii of the five principal spheres.

257. The coordinates ($\xi\eta\zeta\omega\varpi$) of any tangent sphere at the point ($x'y'z'w'v'$) must satisfy

$$\frac{\xi}{(a+k)x'} = \frac{\eta}{(b+k)y'} = \frac{\zeta}{(c+k)z'} = \frac{\omega}{(d+k)w'} = \frac{\varpi}{(e+k)v'}.$$

Hence the equation to the tangent plane at the point $(x'y'z'w'v')$ is

$$(x'x+y'y+z'z+w'w+v'v)\left(\frac{ax'}{r_1}+\frac{by'}{r_2}+\frac{cz'}{r_3}+\frac{dw'}{r_4}+\frac{ev'}{r_5}\right)$$

$$=(ax'x+by'y+cz'z+dw'w+ev'v)\left(\frac{x'}{r_1}+\frac{y'}{r_2}+\frac{z'}{r_3}+\frac{w'}{r_4}+\frac{v'}{r_5}\right), \quad . \quad . \quad . \quad (278)$$

and the equations to the normal at the point $(x'y'z'w'v')$ are

$$\begin{vmatrix} x, & y, & z, & w, & v \\ x', & y', & z', & w', & v' \\ ax', & by', & cz', & dw', & ev' \\ \dfrac{1}{r_1}, & \dfrac{1}{r_2}, & \dfrac{1}{r_3}, & \dfrac{1}{r_4}, & \dfrac{1}{r_5} \end{vmatrix} =0. \quad . \quad . \quad . \quad . \quad . \quad (279)$$

258. The five systems of bitangent spheres are given by:

$$\left.\begin{aligned} \xi=0, &\quad \frac{\eta^2}{b-a}+\frac{\zeta^2}{c-a}+\frac{\omega^2}{d-a}+\frac{\varpi^2}{e-a}=0 \\[1mm] \eta=0, &\quad \frac{\xi^2}{a-b}+\frac{\zeta^2}{c-b}+\frac{\omega^2}{d-b}+\frac{\varpi^2}{e-b}=0 \\[1mm] \zeta=0, &\quad \frac{\xi^2}{a-c}+\frac{\eta^2}{b-c}+\frac{\omega^2}{d-c}+\frac{\varpi^2}{e-c}=0 \\[1mm] \omega=0, &\quad \frac{\xi^2}{a-d}+\frac{\eta^2}{b-d}+\frac{\zeta^2}{c-d}+\frac{\varpi^2}{e-d}=0 \\[1mm] \varpi=0, &\quad \frac{\xi^3}{a-e}+\frac{\eta^2}{b-e}+\frac{\zeta^2}{c-e}+\frac{\omega^2}{d-e}=0 \end{aligned}\right\} ; \quad . \quad . \quad . \quad . \quad (280)$$

and the five systems of focal spheri-quadrics by

$$\left.\begin{aligned} x=0, &\quad \frac{y^2}{b-a}+\frac{z^2}{c-a}+\frac{w^2}{d-a}+\frac{v^2}{e-a}=0 \\[1mm] &\quad\quad y^2+z^2+w^2+v^2=0 \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad (281)$$

and similar equations. These curves have evidently, for their principal circles on the sphere, the circles in which the sphere cuts the other principal spheres of the surface.

259. From the form of the equations of the focal curves, it is clear that any surface represented by the equation

$$\frac{x^2}{\alpha^2+\kappa}+\frac{y^2}{\beta^2+\kappa}+\frac{z^2}{\gamma^2+\kappa}+\frac{w^2}{\delta^2+\kappa}+\frac{v^2}{\epsilon^2+\kappa}=0, \quad . \quad . \quad . \quad . \quad (282)$$

has the same principal spheres and the same focal curves as the surface

$$\frac{x^2}{a^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}+\frac{w^2}{\delta^2}+\frac{v^2}{\epsilon^2}=0 \ ;$$

and subtracting, we have

$$\frac{x^2}{a^2(a^2+\kappa)}+\frac{y^2}{\beta^2(\beta^2+\kappa)}+\frac{z^2}{\gamma^2(\gamma^2+\kappa)}+\frac{w^2}{\delta^2(\delta^2+\kappa)}+\frac{v^2}{\epsilon^2(\epsilon^2+\kappa)}=0,$$

and therefore the spheres whose coordinates are respectively proportional to

$$\frac{x}{a^2}, \ \frac{y}{\beta^2}, \ \frac{z}{\gamma^2}, \ \frac{w}{\delta^2}, \ \frac{v}{\epsilon^2} \ ;$$

and

$$\frac{x}{a^2+\kappa}, \ \frac{y}{\beta^2+\kappa}, \ \frac{z}{\gamma^2+\kappa}, \ \frac{w}{\delta^2+\kappa}, \ \frac{v}{\epsilon^2+\kappa} \ ;$$

cut orthogonally; but these spheres evidently touch the surfaces at a common point; hence, confocal cyclides cut orthogonally. And since the above equation (282) is a cubic in κ, it follows that through any point can be drawn three cyclides confocal with a given cyclide, and these surfaces cut orthogonally.

Or, again, let us determine κ so that (282) represents a cubic cyclide; then we see that three cubic cyclides can, in general, be drawn having the same focal curves as a given quartic cyclide, and these cubic cyclides cut orthogonally. Or, if the given cyclide is a cubic cyclide, three quartic cyclides can be drawn with the same focal curves through any point; and two other cubic cyclides can also be drawn with the same focal curves.

260. Let $(\xi\eta\zeta\omega\varpi)$ be any sphere S; this will cut orthogonally one of the bitangent spheres at the point $(x'y'z'w'v')$ of the surface

$$ax^2+by^2+cz^2+dw^2+ev^2=0,$$

if

$$(a-e)x'\xi+(b-e)y'\eta+(c-e)z'\zeta+(d-e)w'\omega=0.$$

Hence, given any sphere S, a series of bitangent spheres belonging to any system can be drawn, cutting S orthogonally, their points of contact lying on the curve in which the cyclide is cut by the sphere

$$(a-e)\xi x+(b-e)\eta y+(c-e)\zeta z+(d-e)\omega w=0,$$

which may be called the *polar* sphere of S with respect to the cyclide.

There are five such polar spheres for any sphere S, each cutting one of the principal spheres orthogonally, and each one clearly intersects S in points lying on the sphere

$$a\xi x+b\eta y+c\zeta z+d\omega w+e\varpi v=0 \ ;$$

i.e., the five polar spheres of any sphere have with S a common radical plane.

261. If $d=e$, then the equation of the cyclide may be written

$$ax^2 + by^2 + cz^2 = 0.$$

It has clearly two *cnic*-nodes : the points common to (x, y, z) ; and we have seen in § 254 that one of these spheres is imaginary. Also, the surface has three distinct principal spheres ; but any pair of orthogonal spheres cutting (x, y, z) orthogonally may clearly be considered as principal spheres. The case corresponds to that of a quadric of revolution.

The coordinates of a bitangent sphere at the point $(x'y'z'w'v')$ must satisfy the equations

$$\frac{\xi}{(a+k)x'} = \frac{\eta}{(b+k)y'} = \frac{\zeta}{(c+k)z'} = \frac{\omega}{kw'} = \frac{\varpi}{kv'}.$$

There are only three systems of bitangent spheres, viz. :—

$$\left. \begin{array}{l} \xi = 0, \quad \dfrac{\eta^2}{a-b} + \dfrac{\zeta^2}{a-c} + \dfrac{\omega^2+\varpi^2}{a} = 0 \\[2mm] \eta = 0, \quad \dfrac{\xi^2}{b-a} + \dfrac{\zeta^2}{b-c} + \dfrac{\omega^2+\varpi^2}{b} = 0 \\[2mm] \zeta = 0, \quad \dfrac{\xi^2}{c-a} + \dfrac{\eta^2}{c-b} + \dfrac{\omega^2+\varpi^2}{c} = 0 \end{array} \right\} \quad \dots \dots (283)$$

The focal curves on the principal spheres will be circles ; they are given by

$$\left. \begin{array}{l} x = 0, \quad \dfrac{b}{a-b} y^2 + \dfrac{c}{a-c} z^2 = 0 \\[2mm] y = 0, \quad \dfrac{a}{b-a} x^2 + \dfrac{c}{b-c} z^2 = 0 \\[2mm] z = 0, \quad \dfrac{a}{c-a} x^2 + \dfrac{b}{c-b} y^2 = 0 \end{array} \right\} \quad \dots \dots (284)$$

262. Again, if $b=c$, $d=e$, we have seen that the cyclide has four *cnic*-nodes, and one principal sphere x, which is imaginary. There is one system of bitangent spheres, given by

$$\xi = 0, \quad \frac{\eta^2+\zeta^2}{a-b} + \frac{\omega^2+\varpi^2}{a-d} = 0. \quad \dots \dots (285)$$

263. Let us suppose now the radius of one of the principal spheres to be infinite, say r_5 ; the corresponding focal curve is a plane bicircular quartic, its equation being

$$\frac{x^2}{a-e} + \frac{y^2}{b-e} + \frac{z^2}{c-e} + \frac{w^2}{d-e} = 0. \quad \dots \dots (286)$$

And this plane $v=0$ must clearly pass through the centres of the other spheres. Suppose now this curve to represent a circular cubic, then

$$\frac{1}{r_1^2}\frac{1}{a-e}+\frac{1}{r_2^2}\frac{1}{b-e}+\frac{1}{r_3^2}\frac{1}{c-e}+\frac{1}{r_4^2}\frac{1}{d-e}=0. \quad . \quad . \quad . \quad . \quad (287)$$

The curve then passes through the centres of the principal spheres; and since these points will then satisfy the condition of being foci, the imaginary circle at infinity must be a cuspidal edge on the cyclide. A cyclide of this nature is called a Cartesian.

264. If the coefficients of the equation

$$ax^2+by^2+cz^2+dw^2+ev^2=0,$$

are connected by the relation

$$\frac{a}{r_1^2}+\frac{b}{r_2^2}+\frac{c}{r_3^2}+\frac{d}{r_4^2}+\frac{e}{r_5^2}=0,$$

the surface represents a cubic cyclide.

Regarding this as the general cubic cyclide, we see that it passes through the centres of its five principal spheres.

The equation of the tangent plane at the centre of the sphere $x=0$, being

$$\frac{(a-b)}{r_2}y+\frac{(a-c)}{r_3}z+\frac{(a-d)}{r_4}w+\frac{(a-e)}{r_5}v=0 ; \quad . \quad . \quad . \quad . \quad (288)$$

which is clearly parallel to the plane

$$\frac{ax}{r_1}+\frac{by}{r_2}+\frac{cz}{r_3}+\frac{dw}{r_4}+\frac{ev}{r_5}=0.$$

Thus the tangent planes to the cyclide at the centres of its five principal spheres are all parallel, and they may be regarded as the five tangent planes to the cubic from the line at infinity on the surface.

(B.) *General Nodal Cyclide.*—§§ 265–271.

265. We have seen in § 253, that the equation of a cyclide having one node can be expressed in the form

$$ax^2+by^2+cz^2+dw^2=0.$$

The system of reference being three orthogonal spheres (x, y, z) and their two points of intersection (w, v): so that the equation of the absolute is

$$x^2+y^2+z^2=4wv ;$$

the coordinates of the plane at infinity being $\frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r_3}, \frac{1}{\epsilon}, \frac{1}{\epsilon}$, where r_1, r_2, r_3 are the radii of (x, y, z), and ϵ the distance between (w, v). The point $w=0$ is a *cnic*-node.

266. The coordinates $(\xi, \eta, \zeta, \omega, \varpi)$ of any tangent sphere at the point $(x'y'z'w'v')$ must satisfy

$$\frac{\xi}{(a+k)x'} = \frac{\eta}{(b+k)y'} = \frac{\zeta}{(c+k)z'} = \frac{-2\varpi}{dw'-2kv'} = \frac{-2\omega}{-2kw'}.$$

So that the equation of the tangent plane at any point $(x'y'z'w'v')$ is

$$\left(\frac{ax'}{r_1}+\frac{by'}{r_2}+\frac{cz'}{r_3}+\frac{dw'}{\epsilon}\right)(x'x+y'y+z'z-2w'v-2v'w)$$

$$=\left(\frac{x'}{r_1}+\frac{y'}{r_2}+\frac{z'}{r_3}-2\frac{w'+v'}{\epsilon}\right)(ax'x+by'y+cz'z+dw'w); \quad \ldots \quad (289)$$

and the equations of the normal at $(x'y'z'w'v')$ are

$$\left\| \begin{array}{ccccc} x, & y, & z, & -2v, & -2w \\ x', & y', & z', & -2v', & -2w' \\ ax', & by', & cz', & dw', & 0 \\ \dfrac{1}{r_1}, & \dfrac{1}{r_2}, & \dfrac{1}{r_3}, & -\dfrac{2}{\epsilon}, & -\dfrac{2}{\epsilon} \end{array} \right\| = 0. \quad \ldots \quad (290)$$

267. The three systems of bitangent spheres are given by:

$$\left. \begin{array}{l} \xi=0; \quad \dfrac{\eta^2}{b-a}+\dfrac{\zeta^2}{c-a}+\dfrac{4\omega\varpi}{a}-\dfrac{d}{a^2}\omega^2=0 \\[2mm] \eta=0; \quad \dfrac{\xi^2}{a-b}+\dfrac{\zeta^2}{c-b}+\dfrac{4\omega\varpi}{b}-\dfrac{d}{b^2}\omega^2=0 \\[2mm] \zeta=0; \quad \dfrac{\xi^2}{a-c}+\dfrac{\eta^2}{b-c}+\dfrac{4\omega\varpi}{c}-\dfrac{d}{c^2}\omega^2=0 \end{array} \right\} ; \quad \ldots \quad (291)$$

the focal curves are given by:

$$\left. \begin{array}{l} x=0; \quad \dfrac{b}{b-a}y^2+\dfrac{c}{c-a}z^2-\dfrac{d}{a}w^2=0 \\[2mm] \qquad\qquad y^2+z^2-4wv=0 \end{array} \right\} ; \quad \ldots \quad (292)$$

and similar equations. Thus the focal curves are nodal spheri-quadrics on the three principal spheres.

268. From the form of the equations of the focal curves we infer that all surfaces represented by the equation

$$\frac{x^2}{\alpha^2+k}+\frac{y^2}{\beta^2+k}+\frac{z^2}{\gamma^2+k}+\frac{w^2}{\delta^2}=0,$$

are confocal with the surface

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}+\frac{w^2}{\delta^2}=0 \ ;$$

and we infer that given any cyclide having one *cnic*-node, through any point three other cyclides can be drawn having the same node and the same focal curves: and these cyclides cut orthogonally.

We also see, by choosing k so that

$$\frac{1}{\alpha^2+k}\frac{1}{r_1{}^2}+\frac{1}{\beta^2+k}\frac{1}{r_2{}^2}+\frac{1}{\gamma^2+k}\frac{1}{r_3{}^2}+\frac{1}{\delta^2}\frac{1}{\epsilon^2}=0,$$

that three cubic cyclides can be drawn having the same node and focal curves as a given nodal quartic cyclide.

269. If we have $b=c$, we have seen that the equation

$$ax^2+by^2+bz^2+dw^2=0$$

represents a trinodal cyclide, $w=0$ being one node : its equation may also be written

$$(a+b)x^2+dw^2+4bwv=0,$$

or

$$a'x^2+2fwu=0,$$

where u is a sphere passing through the other nodes, but in this case the equation of the absolute will not be so simple. Taking the first form of the equation we see that the system of bitangent spheres is given by

$$\xi=0 \ ; \quad \frac{\eta^2+\zeta^2}{b-a}+\frac{4\omega\varpi}{a}-\frac{d}{a^2}\omega^2=0 \ ; \quad \ldots \ldots \ldots \quad (293)$$

and the corresponding focal curve by

$$x=0 \ ; \quad \frac{4bv}{b-a}-\frac{d}{a}w=0, \ldots \ldots \ldots \ldots \quad (294)$$

which is a circle on the principal sphere $x=0$.

270. If one of the principal spheres, say z, become a plane, the corresponding focal curve is a plane nodal bicircular quartic, and if it pass through the centres of the other two principal spheres, we have

$$\frac{a}{a-c}\frac{1}{r_1{}^2}+\frac{b}{b-c}\frac{1}{r_2{}^2}-\frac{d}{c^2}\frac{1}{\epsilon^2}=0, \quad \ldots \ldots \ldots \quad (295)$$

and if this be satisfied the surface has the circle at infinity for a cuspidal edge ; and so may be called a nodal-Cartesian.

271. If the coefficients in the equation

$$ax^2 + by^2 + cz^2 + dw^2 = 0,$$

are connected by the relation

$$\frac{a}{r_1^2} + \frac{b}{r_2^2} + \frac{c}{r_3^2} + \frac{d}{\epsilon^2} = 0,$$

the surface represents a nodal circular cubic. It clearly passes through the centres of the three principal spheres, and the tangent planes at these points are parallel to the plane

$$\frac{ax}{r_1} + \frac{by}{r_2} + \frac{cz}{r_3} + \frac{dw}{\epsilon} = 0 ;$$

and so the tangent planes at the centres of the principal spheres are the tangent planes drawn from the line at infinity on the surface.

(C.) *Cyclides having a Binode.*—§§ 272–276.

272. The general equation of a cyclide having two principal spheres and a binode can, by § 254, be expressed in the form

$$ax^2 + by^2 + 2hzw = 0 ;$$

the system of reference being the two principal spheres (x, y) and the sphere (z) passing through the node (w), and cutting (x, y) orthogonally in the points (w, v), the equation of the absolute being

$$x^2 + y^2 + z^2 = 4wv,$$

and the coordinates of the plane at infinity $\dfrac{1}{r_1}, \dfrac{1}{r_2}, \dfrac{1}{r_3}, \dfrac{1}{\epsilon}, \dfrac{1}{\epsilon}.$

273. The coordinates $(\xi\eta\zeta\omega\varpi)$ of any tangent sphere at the point $(x'y'z'w'v')$, must satisfy

$$\frac{\xi}{(a+k)x'} = \frac{\eta}{(b+k)y'} = \frac{\zeta}{hw' + kz'} = \frac{-2\varpi}{hz' - 2kv'} = \frac{-2\omega}{-2kw'}.$$

The equation of the tangent plane at the point $(x'y'z'w'v')$ will be

$$\left(\frac{ax'}{r_1} + \frac{by'}{r_2} + \frac{hw'}{r_3} + \frac{hz'}{\epsilon} \right)(x'x + y'y + z'z - 2w'v - 2v'w)$$

$$= \left(\frac{x'}{r_1} + \frac{y'}{r_2} + \frac{z'}{r_3} - 2\frac{w' + v'}{\epsilon} \right)(ax'x + by'y + hw'z + hz'w). \quad . \quad . \quad . \quad (296)$$

The equations of the normals will be

$$
\begin{Vmatrix}
x, & y, & z, & -2v, & -2w \\
x', & y', & z', & -2v', & -2w' \\
ax', & by', & hw' & hz', & 0 \\
\dfrac{1}{r_1}, & \dfrac{1}{r_2}, & \dfrac{1}{r_3}, & -\dfrac{2}{\epsilon}, & -\dfrac{2}{\epsilon}
\end{Vmatrix} = 0. \quad . \quad . \quad . \quad . \quad (297)
$$

274. The two systems of bitangent spheres are given by

$$
\left.
\begin{aligned}
\xi = 0 \; ; & \quad \frac{\eta^2}{b-a} + \frac{4\omega\varpi}{a} - \frac{(a\zeta + h\omega)^2}{a^3} = 0 \\
\eta = 0 \; ; & \quad \frac{\xi^2}{a-b} + \frac{4\omega\varpi}{b} - \frac{(b\zeta + h\omega)^2}{b^3} = 0
\end{aligned}
\right\} \; ; \quad . \quad . \quad . \quad . \quad (298)
$$

and focal curves are given by

$$
\left.
\begin{aligned}
& x = 0 \; ; \quad
\left.
\begin{aligned}
& \frac{b}{b-a}y^2 - \frac{2h}{a}zw - \frac{h^2}{a^2}w^2 = 0 \\
& y^2 + z^2 = 4wv
\end{aligned}
\right\} \\[2ex]
\text{and} \quad & y = 0 \; ; \quad
\left.
\begin{aligned}
& \frac{a}{a-b}x^2 - \frac{2h}{b}zw - \frac{h^2}{b^2}w^2 = 0 \\
& y^2 + z^2 = 4wv
\end{aligned}
\right\}
\end{aligned}
\right\} \quad . \quad . \quad . \quad . \quad (299)
$$

275. Let us suppose the sphere y to be of infinite radius, then the corresponding focal curve will pass through the centre of the principal sphere x, when

$$
\frac{a}{a-b}\frac{1}{r_1^2} - \frac{2h}{b}\frac{1}{\epsilon r_3} - \frac{h^2}{b^2}\frac{1}{\epsilon^2} = 0 \; ; \quad . \quad . \quad . \quad . \quad (300)
$$

and then the surface will represent a Cartesian having a binode.

276. The surface

$$
ax^2 + by^2 + 2hzw = 0
$$

will represent a cubic cyclide, having a binode when

$$
\frac{a}{r_1^2} + \frac{b}{r_2^2} + \frac{2h}{r_3\epsilon} = 0.
$$

In this case the surface passes through the centres of the two principal spheres (x, y), and the tangents at these points are parallel to the plane

$$
\frac{ax}{r_1} + \frac{by}{r_2} + h\frac{z}{\epsilon} + h\frac{w}{r_3} = 0,
$$

and so are the tangent planes drawn through the line at infinity on the surface.

D. *Nodal Cyclides, with One Principal Sphere.*—§§ 277–280.

277. If the cyclide have only one principal sphere, we have seen, in § 255, that its equation may be reduced to the form

$$ax^2 + 2hyz + dw^2 = 0,$$

where x is the principal sphere, w the node, and (y, z) two other spheres cutting x and each other orthogonally, and cutting x in the points (w, v).

The equation of the absolute being

$$x^2 + y^2 + z^2 = 4wv,$$

and the coordinates of the plane at infinity $\dfrac{1}{r_1}, \dfrac{1}{r_2}, \dfrac{1}{r_3}, \dfrac{1}{\epsilon}, \dfrac{1}{\epsilon}$; we have seen that the node is in general a *cnic*-node; but it is a binode if $a = 0$, and a unode if $h = 0$.

278. The coordinates ($\xi\eta\zeta\omega\varpi$) of any tangent sphere at the point ($xyzwv$) must satisfy

$$\frac{\xi}{(a+k)x'} = \frac{\eta}{hz'+ky'} = \frac{\zeta}{hy'+kz'} = \frac{-2\varpi}{dw'-2kv'} = \frac{-2\omega}{-2kw'}.$$

The equation of the tangent plane at ($x'y'z'w'v'$) will be

$$\left(\frac{ax'}{r_1} + \frac{hz'}{r_2} + \frac{hy'}{r_3} + \frac{dw'}{\epsilon} \right)(x'x + y'y + z'z - 2w'v - 2v'w)$$

$$= \left(\frac{x'}{r_1} + \frac{y'}{r_2} + \frac{z'}{r_3} - 2\frac{w'+v'}{\epsilon} \right)(ax'x + hy'z + hz'y + dww'). \quad \dots \quad (301)$$

The equations of the normal at ($x'y'z'w'v'$) are

$$\begin{vmatrix} x, & y, & z, & -2v, & -2w \\ x', & y', & z', & -2v', & -2w' \\ ax', & hz', & hy', & dw', & 0 \\ \dfrac{1}{r_1}, & \dfrac{1}{r_2}, & \dfrac{1}{r_3}, & -\dfrac{2}{\epsilon}, & -\dfrac{2}{\epsilon} \end{vmatrix} = 0. \quad \dots \quad (302)$$

279. The system of bitangent spheres is given by

$$\xi = 0, \quad \frac{a(\eta^2 + \zeta^2) + 2h\eta\zeta}{a^2 - h^2} + \frac{d\omega^2}{a^2} - 4\frac{\omega\varpi}{a} = 0, \quad \dots \quad \dots \quad (303)$$

and the corresponding focal curve is given by,

$$x = 0, \quad \left. \begin{array}{c} \dfrac{4h^2wv}{a^2 - h^2} + \dfrac{ah}{a^2 - h^2}yz + \dfrac{d}{a}w^2 = 0 \\[2mm] y^2 + z^2 = 4wv \end{array} \right\} \quad \dots \quad \dots \quad (304)$$

280. If

$$\frac{a}{r_1{}^2}+\frac{2h}{r_2 r_3}+\frac{d}{\epsilon^2}=0,$$

the surface represented by the equation

$$ax^2+2hyz+dw^2=0,$$

will be a cubic surface, which passes through the centre of the principal sphere, the tangent at which point is parallel to the plane

$$\frac{ax}{r_1}+\frac{hz}{r_2}+\frac{hy}{r_3}+\frac{dw}{\epsilon}=0,$$

and therefore passes through the line on the surface at infinity.

CHAPTER VII.—MISCELLANEOUS THEOREMS.

Equation of a Cyclide referred to Four Spheres Orthogonal to a Principal Sphere.—
§§ 281–285.

281. Let the system of spheres $(xyzwv)$ be such that $v=0$ is a principal sphere of a cyclide ; then its equation must take the form

$$v^2+\phi(xyzw)=0,$$

and if v be orthogonal to the system $(xyzw)$, then the equation of the absolute must also be of the form

$$v^2+f(xyzw)=0.$$

Hence, by subtraction, we see that the equation of a cyclide can always be written in the form

$$ax^2+by^2+cz^2+dw^2+2fyz+2gzx+2hxy+2lxw+2myw+2nzw=0 ; \quad (305)$$

the system of reference being four spheres, points or planes orthogonal to a principal sphere.

Now, the equation of any sphere orthogonal to this same principal sphere is of the form

$$ax+\beta y+\gamma z+\delta w=0, \quad\ldots\ldots\quad (306)$$

where, by § 211, equation (222), α, β, γ, δ must be proportional to the tetrahedral coordinates of the centre of the sphere, referred to the tetrahedron formed by the centres of $(xyzw)$, provided that $(xyzw)$ the coordinates of any point are the powers of the point with respect to the system.

Suppose now that the sphere given by (306) is a bitangent sphere of (305), then clearly we must have

$$\begin{vmatrix} a, & h, & g, & l, & \alpha \\ h, & b, & f, & m, & \beta \\ g, & f, & c, & n, & \gamma \\ l, & m, & n, & d, & \delta \\ \alpha, & \beta, & \gamma, & \delta, & 0 \end{vmatrix} = 0. \quad \ldots \ldots \quad (307)$$

Hence the locus of the centres of all bitangent spheres of the system orthogonal to $v=0$ is the quadric surface given by (307). This surface is called by CASEY the "focal quadric." We see that the focal curve on the principal sphere $v=0$ is the curve of intersection of the sphere with the focal quadric.

282. The surface (305) is a cubic cyclide, if it is satisfied by the coordinates (1, 1, 1, 1) of the plane at infinity; it follows that when this is the case the quadric (307) is touched by the plane at infinity, and, therefore, the focal quadrics of cubic cyclides are paraboloids.

283. If we take for our spheres of reference the other four principal spheres, the equation of the cyclide takes the form

$$ax^2+by^2+cz^2+dw^2=0,$$

and then the focal quadric is given by

$$\frac{\alpha^2}{a}+\frac{\beta^2}{b}+\frac{\gamma^2}{c}+\frac{\delta^2}{w}=0 ;$$

so that the tetrahedron formed by the centres of any four principal spheres of a cyclide is self-conjugate with respect to the focal quadric corresponding to the fifth.

284. In the case of a Cartesian, the focal quadrics must be spheres: hence, if A, B, C, D be the centres of four bitangent spheres which we will take for our spheres of reference, then since the focal quadric must be of the form

$$a^2\beta\gamma+b^2\gamma\alpha+c^2\alpha\beta+a'^2\alpha\delta+b'^2\beta\delta+c'^2\gamma\delta=0,$$

where a, b, c, a', b', c' are the sides of the tetrahedron ABCD, the equation of the cyclide will be

$$\begin{vmatrix} 0, & c^2, & b^2, & a'^2, & x \\ c^2, & 0, & a^2, & b'^2, & y \\ b^2, & a^2, & 0, & c'^2, & z \\ a'^2, & b'^2, & c'^2, & 0, & w \\ x, & y, & z, & w, & 0 \end{vmatrix} = 0 ; \quad \ldots \ldots \quad (308)$$

where $(xyzw)$ are the squares of the tangents from any point P on the surface, to the four bitangent spheres.

285. By § 260 we can draw a series of bitangent spheres belonging to each system orthogonally to any sphere, and the points of contact of each series lie on a sphere which cuts the principal sphere belonging to that series of bitangent spheres orthogonally, and may be called the *polar* sphere of the given sphere with respect to the cyclide. Now let (x, y, z) be any three bitangent spheres cutting a fixed sphere S orthogonally, and let w be the corresponding polar sphere of S, the equation of the cyclide must take the form

$$d^2w^2 + a^2x^2 + b^2y^2 + c^2z^2 - 2bcyz - 2cazx - 2abxy = 0,$$

and the corresponding focal quadric will be

$$2\frac{\delta^2}{d^2} - \frac{\beta\gamma}{bc} - \frac{\gamma\alpha}{ca} - \frac{\alpha\beta}{ab} = 0.$$

Thus the polar of the centre of w with respect to the quadric is the plane passing through the centres of x, y, z.

As a particular case we may suppose S replaced by the centre of the principal sphere, so that (x, y, z) become double tangent planes; the plane $\delta = 0$ is now at an infinite distance, and so we see that the centre of the polar of the centre of the principal sphere with respect to the cyclide is also the centre of the corresponding focal quadric. And also the asymptotic cone of the quadric cuts orthogonally the double tangent cone from the centre of the corresponding principal sphere.

Normals to a Cyclide from any Point. — §§ 286, 287.

286. The problem of drawing normals to a cyclide has been extensively discussed by DARBOUX ('Sur une Classe remarquable de Courbes et de Surfaces Algébriques.' Note XI.). To find the number of normals which can be drawn from any point he proceeds thus :—

Let $(\xi\eta\zeta\omega\varpi)$ be any tangent sphere to the cyclide

$$ax^2 + by^2 + cz^2 + dw^2 + ev^2 = 0,$$

then since

$$\frac{\xi}{(a+k)x} = \frac{\eta}{(b+k)y} = \frac{\zeta}{(c+k)z} = \frac{\omega}{(d+k)w} = \frac{\varpi}{(e+k)v};$$

we have the equations

$$\frac{\xi^2}{a+k} + \frac{\eta^2}{b+k} + \frac{\zeta^2}{c+k} + \frac{\omega^2}{d+k} + \frac{\varpi^2}{e+k} = 0,$$

and

$$\frac{\xi^2}{(a+k)^2} + \frac{\eta^2}{(b+k)^2} + \frac{\zeta^2}{(c+k)^2} + \frac{\omega^2}{(d+k)^2} + \frac{\varpi^2}{(e+k)^2} = 0.$$

Now suppose

$$\frac{\xi}{\alpha+\mu\alpha'}=\frac{\eta}{\beta+\mu\beta'}=\frac{\zeta}{\gamma+\mu\gamma'}=\frac{\omega}{\delta+\mu\delta'}=\frac{\varpi}{\epsilon+\mu\epsilon'},$$

then substituting and eliminating k, we have clearly an equation of the twelfth degree in μ; hence we infer that twelve spheres can be drawn through the circle common to the spheres $(\alpha\beta\gamma\delta\epsilon)$ $(\alpha'\beta'\gamma'\delta'\epsilon')$ to touch the cyclide.

We may deduce from this that through any straight line can be drawn twelve tangent planes to the cyclide, or a cyclide is in general of the twelfth class.

And, again, taking ξ, η, ζ, ω, ϖ proportional to $x+\dfrac{\mu}{r_1}$, $y+\dfrac{\mu}{r_2}$, $z+\dfrac{\mu}{r_3}$, $w+\dfrac{\mu}{r_4}$, $v+\dfrac{\mu}{r_5}$ respectively, we see that twelve spheres can be drawn to touch the cyclide, having their centres coincident with the point $(xyzwv)$. Hence twelve normals can be drawn to the cyclide from any point.

287. In § 257, equations (279) give the normal at the point $(x'y'z'w'v')$ of the cyclide

$$ax^2+by^2+cz^2+dw^2+ev^2=0,$$

and we infer that the feet of the normals from the point $(x'y'z'w'v')$ lie on the cubic surfaces :

$$\left\| \begin{array}{ccccc} ax, & by, & cz, & dw, & ev \\ x, & y, & z, & w, & v \\ x', & y', & z', & w', & v' \\ \dfrac{1}{r_1}, & \dfrac{1}{r_2}, & \dfrac{1}{r_3}, & \dfrac{1}{r_4}, & \dfrac{1}{r_5} \end{array} \right\| =0 \quad . \quad . \quad . \quad . \quad . \quad (309)$$

Now, $\psi=0$ being the equation of the absolute referred to any system of spheres, ϕ_1, ϕ_2, ϕ_3 any three cyclides; let us form the discriminating quintic of

$$\lambda\phi_1+\mu\phi_2+\nu\phi_3+\rho\psi=0 ;$$

then we can so choose the ratios $\lambda : \mu : \nu : \rho$ that three of the roots of this quintic shall be zero; and thus the equation will represent two spheres. To determine the ratios $\dfrac{\lambda}{\rho}, \dfrac{\mu}{\rho}, \dfrac{\nu}{\rho}$ we obtain three equations of the fifth, fourth, and third degrees respectively. Hence through the common points of three cyclides a pair of spheres can be drawn in sixty ways.

Let us suppose now that certain of the feet of the twelve normals from the point $(x'y'z'w'v')$ lie on the sphere

$$\xi x+\eta y+\zeta z+\omega w+\varpi v=0,$$

then the rest must lie on another sphere which we may denote by

$$\alpha x+\beta y+\gamma z+\delta w+\epsilon v=0,$$

and then it must be possible so to choose k that the equation

$$k(ax^2+by^2+cz^2+dw^2+ev^2)=(\alpha x+\beta y+\gamma z+\delta w+\epsilon v)(\xi x+\eta y+\zeta z+\omega w+\varpi v), \quad (310)$$

shall be identical with any one of the cubic cyclides given by (308).   We have at once then

$$\frac{\alpha\xi}{a}=\frac{\beta\eta}{b}=\frac{\nu\zeta}{c}=\frac{\delta\omega}{d}=\frac{\epsilon\varpi}{e}=k.$$

Consequently through the feet of the twelve normals from $(x'y'z'w'v')$ can be drawn two spheres, such that $(\xi\eta\zeta\omega\varpi)$ being the coordinates of one of them, $\frac{a}{\xi}, \frac{b}{\eta}, \frac{c}{\zeta}, \frac{d}{\omega}, \frac{e}{\varpi}$ must be proportional to the coordinates of the other.

Also equation (310) must represent a cubic: hence we must have

$$\frac{a}{r_1^2}+\frac{b}{r_2^2}+\frac{c}{r_3^2}+\frac{d}{r_4^2}+\frac{e}{r_5^2}=\left(\frac{\xi}{r_1}+\frac{\eta}{r_2}+\frac{\zeta}{r_3}+\frac{\omega}{r_4}+\frac{\varpi}{r_5}\right)\left(\frac{a}{\xi r_1}+\frac{b}{\eta r_2}+\frac{c}{\zeta r_3}+\frac{d}{\omega r_4}+\frac{e}{\varpi r_5}\right).$$

So that if the given surface is a cubic cyclide, one of these spheres on which the feet of the normals lie must be of infinite radius.